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NOTE ON THE BEHAVIOUR OF THE EÖTVÖS GRAVITY BALANCE IN FLUCTUATING GRAVITATIONAL FIELDS

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ABSTRACT. In this note attention is directed to the semi-diurnal variation of gravity at a point on the earth's surface, due to lunar attraction and recently measured by Loomis. It is shown that this temporal variation of g is much larger than the spatial differences measured by the Eötvös gravity balance, but that it produces no effect on the balance. This constitutes an experimental proof of the power of the Eötvös instrument to discriminate between space and time changes of terrestrial gravitation.

IN my presidential address to the Society* I remarked that the Eötvös torsion balance had so far been denied the opportunity of demonstrating its chief virtue, namely, its capacity for differentiating between temporal and spatial variations of the earth's gravitational field, by responding to the latter and not to the former. This remark was made without due consideration of the facts, and it is not true. I had forgotten that it has recently been established experimentally that there is a semi-diurnal fluctuation of g due to the gravitational attraction of the moon. Such variations have been implied, of course, ever since the origin of the tides was recognized, and G. H. Darwin calculated the effect more than fifty years ago. But it was not until 1931, when Loomis published† the results of his comparisons of Shortt clocks with quartz oscillators, and E. W. Brown and D. Brouwer‡ analysed these records, that the effect of the moon upon the period of a pendulum, and consequently upon g, was definitely measured. The results are of such significance in relation to practice with the Eötvös balance, and prove for it such a tacit triumph in its power of discrimination, that they deserve to be quoted in some detail.

The results of Loomis's comparisons show that the error of the Shortt clock,

* *Proc. Phys. Soc.* **46**, 1 (1934).

† *M.N.R.A.S.* **91**, 569 (1931).

‡ *M.N.R.A.S.* **91**, 575 (1931).

as compared with the quartz oscillator which is independent of gravity, is given by the expression $-0{\cdot}00015 \sin 2\phi$ sec., where ϕ is the difference between the longitude of the moon and that of the experimental station (at Tuxedo Park). The error is thus approximately semi-diurnal and of amplitude 0·00015 sec. Taking into account the dependence of the pendulum-period upon gravity we find that the corresponding small variation Δg of g is given by

$$\Delta g/g = -4{\cdot}3 \times 10^{-8} \cos 2\phi,$$

which indicates a fluctuation of Δg of half-lunar-day period and ranging from $-4{\cdot}3 \times 10^{-8} g$ when the moon is in the local meridian (or 180° from it) to $+4{\cdot}3 \times 10^{-8} g$ when the moon is rising or setting. These values are for the latitude of Tuxedo (41° N.) and agree with the general formula derived from G. H. Darwin's theoretical investigation, namely

$$\Delta g/g = -7{\cdot}7 \times 10^{-8} \cos^2\lambda \cos 2\phi,$$

where λ is the latitude. At the equator the effect will be a maximum, and there the total change of g during the six lunar hours above mentioned will be $2 \times 7{\cdot}7 \times 10^{-8} g = 1{\cdot}5 \times 10^{-7} g$ approximately. Temporal fluctuations of this order of magnitude have certainly been occurring regularly ever since Eötvös initiated field practice with his balance, yet there is no record of their having produced any noticeable effect, even though a complete set of observations at a station often itself occupied six hours.

It is interesting to compare the relative magnitudes of the spatial variations which the balance has measured and the temporal variations to which it has remained irresponsive. We may take the time variation of g for the average latitude as, say, $10^{-7} g$, or 10^{-4} cm. sec.$^{-2}$ Compared with the magnetic diurnal variation this is relatively small*, but it bears a large ratio to the spatial differences actuating the Eötvös instrument. A good instrument of this type is sensitive enough to indicate a gravity-gradient of 1 Eötvös unit, or 10^{-9} sec.$^{-2}$, and its full scale seldom corresponds to more than a few hundred Eötvös units, say, 5×10^{-7} sec.$^{-2}$ The balance, in effect, measures the variation of g over a horizontal distance equal to the difference of level of the masses constituting its beam, and this is of the order 50 cm. Thus it will just indicate a change of g amounting to 5×10^{-8} cm. sec.$^{-2}$, and give full-scale deflection for $2{\cdot}5 \times 10^{-5}$ cm. sec.$^{-2}$ The temporal variation, attributable to lunar action, bears to these the ratios 2×10^3 and 4 respectively. It appears then that the Eötvös torsion balance has continued calmly and faithfully to record spatial variations of gravity only, while subjected also to temporal changes two thousand times as large as the least effect it could show, and four times the equivalent of its maximum recording-capacity. Surely a convincing practical proof of the virtue of the balance in discriminating between effects in space and time.

The explanation, of course, lies in the fact of the close approximation to spatial uniformity of the lunar gravitational field over the space occupied by the instrument, combined with its small value relative to the terrestrial field. We may thus confidently anticipate that the magnetic analogue of the gravity torsion

* The diurnal variation of H in London, for example, is of the order $2 \times 10^{-3} H$.

balance would be similarly immune from the effects of large-scale temporal changes of the terrestrial magnetic field, such as the diurnal fluctuations. It is true that the experimental proof of immunity does not extend precisely by implication to magnetic storms, for gravitational storms are rare events if we exclude the inappreciable effects of meteors. But there are no grounds for doubting that most magnetic storms, being of a widespread character, would also be ineffective in disturbing a magnetic torsion balance based on the same principles as the Eötvös gravity instrument.

DISCUSSION

Dr H. Shaw. It is interesting to note, in support of Prof. Rankine's paper, that when on occasions we have observed the torsion balance at hourly intervals continuously on the same station for several weeks, no temporal changes corresponding to the diurnal variations referred to, have been indicated by this instrument. The reason, as is explained in the paper, is that the lunar effect, which is of a temporal nature, gives rise to a practically uniform raising or lowering of the force of gravity over a large area, so that the values of the gravity-gradient and the differential curvature, which are measured by the torsion balance, are comparatively unaffected. In the case of very small and local disturbances, however, these give rise to a local deformation of the gravity surface which will be detected and measured by the torsion balance. Similarly it is reasonable to suppose that a magnetic instrument, such as the one referred to by Prof. Rankine, would not respond to extensive magnetic storms, but would be sensitive to small local disturbances, and so may be expected to be most useful in distinguishing between these two types of disturbances.

536.412

ON THE TEMPERATURE VARIATION OF THE ORTHOBARIC DENSITY OF UNASSOCIATED LIQUIDS

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ABSTRACT. A formula connecting the orthobaric density of a liquid and its temperature is developed in the form

$$\rho = 2\rho_c\,[A\,(1-m)^{0\cdot3} + (1-\tfrac{1}{2}m)],$$

where m is reduced temperature and A is a constant which varies slightly from liquid to liquid, and may be taken to have a mean value 0·91. The formula is a long-range one, and has been tested for thirty pure organic substances. It has been applied to the evaluation of expansion coefficients, and to show the manner in which free and total molecular surface energies vary with temperature.

§ 1. DEVELOPMENT OF THE FORMULA

Some time ago, in a discussion on a paper by Dr Griffiths and Mr Awbery*, it was pointed out by one of us that a simple long-range formula could easily be established which should represent with very high accuracy the relation between orthobaric density and temperature for an unassociated liquid. In this paper we have worked out the constants of the equation for the thirty pure liquids whose pressure-volume-temperature relations have been studied by Prof. S. Young, and discussed in a paper which has become classic†. It is hardly necessary here to stress the importance of obtaining figures giving the thermal properties of liquids and vapours with a high degree of precision, or of developing equations which shall accurately represent these results. The present paper is a contribution towards the end last named.

Some years ago Macleod‡ proposed the equation

$\gamma, \rho_l, \rho_v, C$

$$\gamma = C\,(\rho_l - \rho_v)^4 \qquad \ldots\ldots(1),$$

connecting surface tension at any temperature with the corresponding values of the orthobaric densities of the liquid and its vapour. It has also been shown by one of us (A. F.) that the constant C may be expressed§ in terms of the critical constants. If we combine this equation with the power law‖ connecting surface tension and temperature

$$\gamma = A\,(1-m)^n \qquad \ldots\ldots(2),$$

* *Proc. Phys. Soc.* **44**, 121 (1932).
† *Proc. R. Dub. Soc.* **12**, 374 (1910).
‡ *Trans. Farad. Soc.* **19**, 38 (1923).
§ *Trans. Farad. Soc.* **19**, 17 (1923).
‖ Van der Waals, *Z. f. Phys. Chem.* **13**, 716 (1894); A. Ferguson, *Phil. Mag.* **31**, 37 (1916).

where m is the reduced absolute temperature, and n a number which varies slightly from liquid to liquid, but which may be taken as equal to 1·2, we easily obtain m, n

$$\rho_l - \rho_v = 4\rho_c (1 - m)^{0·3} \quad \ldots\ldots(3).$$

The constant of the equation is evaluated as $4\rho_c$ from the consideration that, when m is zero, ρ_v may be neglected, and ρ_l, the liquid density, may be taken as about four times the critical density.

The law of rectilinear diameters gives us

$$\rho_l + \rho_v = a - bm = 4\rho_c - 2\rho_c m \quad \ldots\ldots(4),$$

from the consideration just mentioned, together with the fact that when $m = 1$, $\rho_l = \rho_v = \rho_c$.

Hence, adding, we find

$$\rho_l = 2\rho_c [(1 - m)^{0·3} + (1 - 0·5\, m)] \quad \ldots\ldots(5).$$

It is noteworthy that, if we may assume the constancy of the indices in Macleod's law and the power law, we have here a general relation between reduced density and reduced temperature which contains no constants depending on the properties of any one liquid. A similar equation may be obtained, of course, for the density of the saturated vapour.

Restricting ourselves to a consideration of the liquid density, and therefore dropping the subscript l, we may write equation (5) as

$$\rho/2\rho_c - (1 - 0·5\, m) = (1 - m)^s \quad \ldots\ldots(6),$$

or

$$\log [\rho/2\rho_c - (1 - 0·5\, m)] = s \log (1 - m) \quad \ldots\ldots(7),$$

where, for generality, we put s for the constant 0·3. A logarithmic plot should therefore give us a straight line through the origin, whose slope will determine s. The thirty pure substances mentioned, for which Young gives experimental values of ρ and m over their whole range of existence, have been tested in this way. Very satisfactory linear plots are obtained, but in no case does the line pass through the origin. If therefore we put s

$$\rho/2\rho_c - (1 - 0·5\, m) = X, \quad (1 - m) = Y \quad \ldots\ldots(8), \qquad X, Y$$

our line becomes

$$\log Y = s \log X + \log A \quad \ldots\ldots(9),$$

where A is determined from the intercept, and our final equation connecting ρ and m is

$$\rho = 2\rho_c [A (1 - m)^s + (1 - 0·5\, m)] \quad \ldots\ldots(10).$$

In table 1 we show a conspectus of the results, giving the values of A and of s. It is noteworthy that the only serious deviations from the value 0·3 for the index s are those shown by the alcohols—typical associated liquids. A varies very little from liquid to liquid, and a reasonably satisfactory general formula for the temperature variation of the density of any unassociated liquid is

$$\rho = 2\rho_c [0·91 (1 - m)^{0·3} + (1 - 0·5\, m)] \quad \ldots\ldots(11),$$

where mean values of A and of s are employed. If high accuracy is desired, account must be taken of the variation of A and of s from liquid to liquid.

Table 1. Values of A and s

Substance	A	s	Substance	A	s
Benzene	0·8790	0·3003	Normal hexane	0·8966	0·2938
Fluorobenzene	0·8831	0·2986	Normal heptane	0·9268	0·3000
Chlorobenzene	0·8995	0·3000	Normal octane	0·9473	0·2934
Bromobenzene	0·9033	0·2934	Isopentane	0·8728	0·3035
Iodobenzene	0·9103	0·3023	Di-isobutyl	0·9020	0·2837
Methyl formate	0·9192	0·3000	Di-isopropyl	0·8510	0·2798
Ethyl formate	0·9294	0·3035	Hexamethylene	0·8586	0·2957
Propyl formate	0·9389	0·3063	Acetic acid	0·9109	0·2914
Ethyl acetate	0·9512	0·3119	Carbon tetrachloride	0·8668	0·3000
Propyl acetate	0·9750	0·3113	Stannic chloride	0·9242	0·3146
Methyl acetate	0·9428	0·3050	Ethyl ether	0·9036	0·2944
Methyl propionate	0·9517	0·3094	Methyl alcohol	0·8855	0·2120 *
Ethyl propionate	0·9815	0·3161	Ethyl alcohol	0·8915	0·2068 *
Methyl butyrate	0·9583	0·3076	Propyl alcohol	0·8772	0·2048 *
Methyl isobutyrate	0·9482	0·3063			
Normal pentane	0·8764	0·2902	Mean values	0·9113	0·3001

* Excluded from mean value.

b Obviously, the common assumption that $v_c = 4b$, where b is the constant of characteristic equations of the type

$$(p + \omega)(v - b) = R\theta,$$

—an assumption which we used in developing equation (10)—is not close enough. If we put m equal to 0 in equation (11) we find that

$$\rho = 2\rho_c\,(1{\cdot}91) \quad \text{or} \quad v_c = 3{\cdot}82\,b.$$

If we use equation (10) with the values of A and s appropriate to the particular liquid, the agreement with the experimental values is remarkably close. Table 2 shows the agreement between observed and calculated values for one of the sub-

Table 2. Methyl propionate

Temperature (° C.)	ρ observed	ρ calculated	Temperature (° C.)	ρ observed	ρ calculated
0	0·9387	0·9393	200	0·6445	0·6448
20	0·9151	0·9158	220	0·5938	0·5952
40	0·8912	0·8912	230	0·5635	0·5662
60	0·8665	0·8666	240	0·5220	0·5302
80	0·8408	0·8408	245	0·4976	0·5057
100	0·8137	0·8137	250	0·4665	0·4753
120	0·7852	0·7852	253	0·4401	0·4573
140	0·7553	0·7544	256	0·3982	0·4061
160	0·7221	0·7216	257·4 (Critical)	0·3124	0·3124
180	0·6856	0·6858			

stances—methyl propionate—selected at random. Figures 1 and 2 show respectively the straight line from which the constants are deduced and the march of the calculated and the observed values for fluorobenzene. It will be seen that, save for a limited region just below the critical temperature, observed and calculated

values agree within the limits of experimental error. At the critical point itself, the formula is adjusted to give exact agreement, and moreover $d\rho/dm$ is, as it should be, infinite.

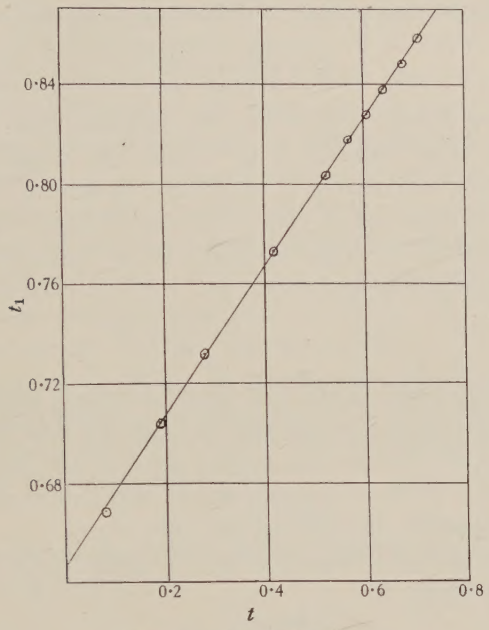

Figure 1. Fluorobenzene.

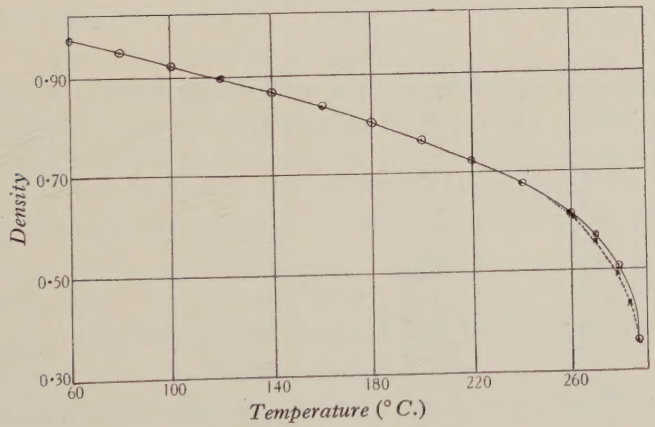

Figure 2. Fluorobenzene.

§ 2. APPLICATIONS

A long-range formula which possesses any pretension to accuracy will usually subsume a number of formulae of restricted application and should, indeed, indicate the reasons for the shortness of the range over which such formulae are valid. Equations (10) and (11) provide some interesting illustrations in the field of heat and of capillarity, one or two of which we proceed to discuss.

(i) Thorpe and Rucker suggested a number of years ago* that the critical temperature of an unassociated liquid might be determined from two observations of the density of the liquid, made at two not very widely separated temperatures (usually 0° C. and the boiling point). The formula proposed was

$$\theta_c = \frac{\theta_2\rho_1 - \theta_1\rho_2}{B(\rho_1 - \rho_2)} \qquad \text{.....(12),}$$

B where B is a constant whose mean value was taken as 1·99. The value of B was calculated for twenty-four substances for which the required data are known, and the extreme range of variation of B was from 1·93 to 2·03. Obviously, therefore, the formula cannot be expected to give very exact results. The boiling point of a normal liquid may be taken, very approximately, as 0·65 times its critical temperature, so that, employing reduced temperatures, and writing equation (12) as

$$B = 1{\cdot}99 = \frac{m_2\rho_1 - m_1\rho_2}{\rho_1 - \rho_2} \qquad \text{.....(13),}$$

δ we may put $m_2 = 0{\cdot}65$, and $m_1 = 0{\cdot}65 - \delta$, where δ is fairly small. If we now, using equation (11), calculate $(m_2\rho_1 - m_1\rho_2)$ and $(\rho_1 - \rho_2)$, we find that their ratio is independent of δ provided that we can neglect squares and higher powers of δ in comparison with unity. If the two temperatures chosen are close enough to make this assumption permissible, the value of the ratio is 1·9.

(ii) Thatte† has proposed an equation for the coefficient of expansion of a van-der-Waals fluid in the form

α, θ_c, θ

$$\alpha = \frac{1}{v}\frac{\partial v}{\partial\theta} = \frac{1}{2{\cdot}5\,\theta_c - 2\theta} \qquad \text{.....(14).}$$

We may write

$$\alpha = -\frac{1}{\rho}\frac{\partial\rho}{\partial\theta} = -\frac{1}{\rho\theta_c}\frac{\partial\rho}{\partial m} \qquad \text{.....(15),}$$

in terms of reduced temperatures. The differential coefficient $\partial\rho/\partial m$ refers to orthobaric conditions, whilst that in equation (14) is taken at constant pressure, but the distinction is unimportant for the small values of m to which the argument is confined. If, taking equation (10) as basis, we evaluate the right-hand side of (15), writing it in the reciprocal form of (14), we find that, approximately,

$$\alpha = -\frac{1}{\rho\theta_c}\frac{\partial\rho}{\partial m} = 1\Big/\left\{\frac{A+1}{As+\frac{1}{2}}\theta_c - \frac{A+1}{As+\frac{1}{2}}\left(\frac{As+\frac{1}{2}}{A+1} + \frac{As(1-s)}{As+\frac{1}{2}}\right)\theta\right\}.$$

If, as a first approximation, we put $A = 1$, $s = 0{\cdot}3$ we find that

$$\alpha = 1/(2{\cdot}5\,\theta_c - 1{\cdot}66\theta) \qquad \text{.....(16).}$$

The closer approximation involved in giving A its mean value 0·91, leads to

$$\alpha = \frac{1}{2{\cdot}47\,\theta_c - 1{\cdot}61\,\theta} \qquad \text{.....(17).}$$

The agreement with Thatte's formula is as close as can be expected, and the simple analysis shows clearly enough the conditions under which formulae of the type (12) or (14) hold good. Apart from the physical fact that the constant A does vary slightly from liquid to liquid—a fact which in itself sets a limit to the accuracy

* *Trans. Chem. Soc.* **45**, 135 (1884); S. Young, *Stoichiometry*, p. 171.
† *Phil. Mag.* **7**, 887 (1929).

of all such formulae—it will be apparent that the simplicity of the equations depends on the possibility of an expansion in which either the reduced temperature, or the difference of two reduced temperatures, is small.

(iii) Reverting to Thorpe and Rucker's problem of the calculation of a critical temperature from a knowledge of the density of a liquid at two temperatures which are well below the critical value, it is evident that the possibility of applying a simple formula is contingent on the temperatures being fairly close together. This suggests the use of a knowledge of the coefficient of expansion and of equation (17) to calculate an approximate value of the critical temperature. Failing this, Thorpe and Rucker's formula would give better results if the two temperatures at which the densities are measured were less widely separated.

(iv) It is desirable to be able to calculate an approximate value of the critical density of a fluid whose critical temperature is known. This is possible, to the limit of accuracy imposed by the variation of A in equation (10), from a knowledge of one value of the density, if we use the mean value of A given in equation (11). If, for simplicity, we choose a definite convenient value for m, say $m = 0{\cdot}6$, and determine the density of the liquid at the corresponding centigrade temperature, we have, putting $m = 0{\cdot}6$ in equation (11),

$$\rho_c = 0{\cdot}3592\rho \qquad \text{.....(18).}$$

The accuracy with which the critical density may be calculated from this simple formula is shown in table 3 below.

Table 3

Substance	Temperature (°C.)	ρ_t	ρ_c calculated	ρ_c observed
Benzene	63·7	0·832	0·298	0·304
Iodobenzene	159·6	1·620	0·581	0·581
Bromobenzene	129·0	1·345	0·483	0·485
Chlorobenzene	106·3	1·012	0·364	0·365
Fluorobenzene	62·7	0·971	0·349	0·354
Pentane	9·1	0·637	0·229	0·232
Hexane	31·6	0·649	0·233	0·232
Heptane	50·1	0·657	0·236	0·234
Octane	68·5	0·662	0·238	0·233
Isopentane	3·4	0·626	0·224	0·234
Hexamethylene	58·8	0·741	0·266	0·273
Di-isobutyl	56·8	0·662	0·237	0·236
Di-isopropyl	27·1	0·655	0·235	0·241
Ethyl acetate	40·8	0·875	0·314	0·308
Propyl acetate	56·5	0·848	0·304	0·296
Methyl acetate	33·4	0·916	0·329	0·325
Ethyl propionate	53·5	0·852	0·306	0·296
Methyl propionate	45·2	0·885	0·318	0·312
Methyl butyrate	59·5	0·854	0·307	0·300
Methyl isobutyrate	51·3	0·853	0·306	0·301
Acetic acid	83·7	0·979	0·352	0·351
Stannic chloride	82·0	2·066	0·742	0·742
Ethyl formate	31·9	0·907	0·326	0·323
Methyl formate	19·2	0·976	0·350	0·349
Propyl formate	49·6	0·870	0·313	0·309
Ethyl ether	7·0	0·728	0·262	0·262
Carbon tetrachloride	60·6	1·515	0·544	0·558

(v) The formula may be used to establish the manner of the temperature variation of certain capillary constants of importance in physical chemistry. *Free* molecular surface energy e and *total* molecular surface energy E are defined by the equations

e, E

$$e = \gamma (M/\rho)^{\frac{2}{3}} \quad \text{and} \quad E = \lambda (M/\rho)^{\frac{2}{3}} \qquad \text{.....(19)}$$

respectively, where M is the molecular weight of the liquid, γ its surface tension (or free surface energy), and λ its total surface energy. λ and γ are connected by the well-known relation

M, γ, λ

$$\gamma = \lambda + \theta \, \partial\gamma/\partial\theta \qquad \text{.....(20).}$$

If we assume the power law (2) connecting surface tension and temperature, the equation which gives the variation of λ with temperature is, as was pointed out by Porter*,

$$\lambda = A (1 - m)^{n-1} [1 + (n - 1) m] \qquad \text{.....(21),}$$

where we may put $n = 1{\cdot}2$.

Combining these equations with the equation representing the march of density with temperature, which for convenience we shall put in its simplest form

$$\rho = 2\rho_c [(1 - m)^{0{\cdot}3} + (1 - \tfrac{1}{2}m)],$$

we easily find that

$$e = K \frac{(1 - m)^{1{\cdot}2}}{[(1 - m)^{0{\cdot}3} + (1 - \frac{1}{2}m)]^{\frac{2}{3}}} \qquad \text{.....(22),}$$

and

$$E = \frac{K_1 (1 - m)^{0{\cdot}2} (1 + 0{\cdot}2\, m)}{[(1 - m)^{0{\cdot}3} + (1 - 0{\cdot}5\, m)]^{\frac{2}{3}}} \qquad \text{.....(23),}$$

where K and K_1 are constants which may readily be evaluated in terms of the molecular weight and the critical constants.

The most direct way of exhibiting the variation of e and E with temperature is to work out the values of the multipliers of K and K_1 for different values of m and to exhibit the results graphically. It is common to assume that E is independent of temperature, and Bennett and Mitchell† have calculated values of E for a large number of substances, and obtained results which are deemed to support this conclusion. If we put

$$E = K_1 F, \text{ where } F = (1 - m)^{0{\cdot}3} (1 + 0{\cdot}2m)/[(1 - m)^{0{\cdot}3} + (1 - 0{\cdot}5m)]^{\frac{2}{3}},$$

we find, on calculating F for different values of m, the results given in table 4.

Table 4

m	0·0	0·3	0·5	0·6	0·7	0·8	0·9	0·99	1·00
F	0·630	0·680	0·711	0·724	0·735	0·738	0·720	0·575	0·000

Figure 3 shows the variation in F, and therefore the variation of the total molecular surface energy E with temperature. It will be seen that there is a slight but definite maximum at about $m = 0{\cdot}79$.

* *Trans. Farad. Soc.* **17**, 1 (1922).
† *Z. f. Phys. Chem.* **84**, 475 (1913).

Now take Bennett and Mitchell's figures for, say, chlorobenzene. The critical temperature of chlorobenzene is 633°, the figures are given in table 5 and cover the range $m = 0{\cdot}67$ to $m = 0{\cdot}85$.

Table 5. Chlorobenzene

θ	423	433	443	453	463	473	483	493
E	1461	1462	1467	1469	1472	1470	1474	1471

θ	513	523	533	543	553	563
E	1468	1467	1467	1468	1461	1455

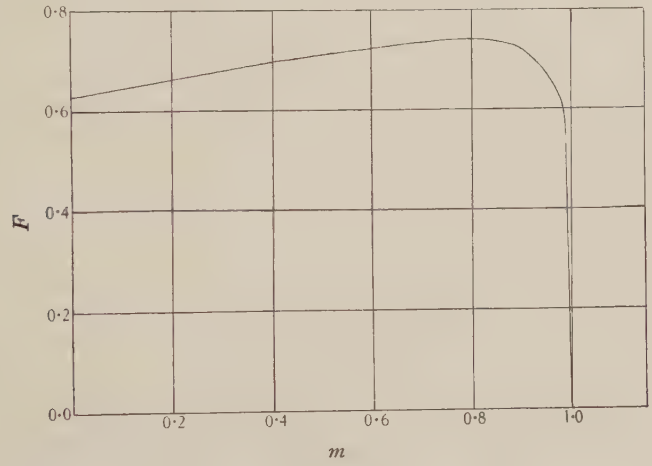

Figure 3.

These figures are usually quoted in support of the constancy of E, and the slight maximum which they show has in consequence been overlooked. Nevertheless the maximum is there, and occurs at about 483°, for which $m = 0{\cdot}76$. This is in close agreement with the maximum given by the theoretical curve.

If we plot in similar fashion the multiplier of K in the expression for e, the resulting curve is indistinguishable from a straight line, except in a limited region near to the critical point. This is in accordance with the experimental results obtained by Ramsay and Shields.

We propose, in a future communication, to discuss the temperature variation of orthobaric vapour density, and the relation of saturated vapour pressure to temperature.

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CRYSTAL STRUCTURE AND ORIENTATION IN ZINC-OXIDE FILMS

By G. I. FINCH and A. G. QUARRELL, Imperial College of Science and Technology

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ABSTRACT. A new type of electron-diffraction camera is described incorporating means for greatly increasing the accuracy hitherto obtainable in electron-diffraction analysis. Partially and completely oxidized zinc films have been examined by transmission. It has been found that the normal type of zinc oxide is formed by the oxidation of zinc via a zinc oxide which is basally pseudomorphic with the zinc. The corrosion-resisting properties of zinc appear to be due, in the main, to a protective coating of such pseudomorphic zinc oxide.

§ 1. INTRODUCTION

The well-known phenomenon of pseudomorphism in crystal growth extends to all three dimensions. It has recently been shown, however, that when formed under suitable conditions thin films of aluminium on platinum, of zinc oxide on zinc, and of magnesium oxide on magnesium, possess abnormal crystal structures due to pseudomorphism confined to two dimensions*. For example, aluminium, normally of face-centred cubic structure (a = 4·05 Å.), when deposited as a sufficiently thin layer on normal face-centred cubic platinum (a = 3·91 Å.) acquired a face-centred tetragonal structure with basal axis 3·90 Å. and major axis 4·02 Å. Thus, within the then limits of experimental error, the aluminium was in its basal dimensions pseudomorphic with the platinum substrate. A similar effect was observed in the case of a thin film of zinc oxide on zinc (triangular close-packed lattice, hexagonal system; a = 2·65 Å., c = 1·95). In this case, whilst the basal dimensions of the zinc oxide were similar to those of the zinc substrate, the major axis had increased from 5·18 Å., the value for the normal form of zinc oxide, to 6·79 Å. Hence the volume of the cell of abnormal zinc oxide basally pseudomorphic with normal zinc was nearly equal to that of the normal zinc-oxide cell. The new phenomenon thus brought to light may conveniently be termed basal-plane pseudomorphism.

In the case of aluminium exhibiting basal-plane pseudomorphism with normal face-centred cubic platinum, it so happened that, owing to the doubling or trebling of certain rings, the diffraction patterns obtained were such as to establish clearly the face-centred tetragonal structure of such aluminium. On the other hand, with zinc oxide formed on zinc the type of crystal structure was found to remain unchanged. The accuracy attainable in the determination of lattice dimensions by

* *Proc. R.S.* A **141**, 398 (1933).

existing methods of electron diffraction, however, proved to be amply sufficient for the purpose of establishing the occurrence of basal-plane pseudomorphism in this case also. Nevertheless, it was evident that in order to carry out effectively a critical search for a structure intermediate between one exhibiting full basal-plane pseudomorphism and the normal structure, it would be necessary to evolve means whereby the accuracy of electron-diffraction measurements might be greatly increased.

The accuracy of the determination of crystal-lattice constants by electron diffraction has hitherto been limited by various sources of error, chief of which is that due to the lack of precision attendant upon the measurement of high voltages. In addition, in the case of diffraction of an unfocused beam of electrons at grazing incidence, an uncertainty as to the effective camera length is introduced by the fact that the beam is divergent and the rings are correspondingly blurred. In what follows an account is given of an electron-diffraction camera which incorporates devices enabling these and other sources of error to be eliminated to such a degree that the accuracy attainable is comparable with that of the most precise X-ray methods of crystal analysis. With this camera a detailed examination of the structure of zinc—zinc-oxide films has been carried out, and the great gain in accuracy achieved has led, inter alia, to a new view of the causes affecting the distribution of the intensities of the rings in electron-diffraction patterns.

§ 2. EXPERIMENTAL

Owing to the sources of error outlined above, and notwithstanding many claims to the contrary, an accuracy greater than $\pm$ 2 per cent, when medium-speed electrons (above, say, 20 kV.) are employed for the determinations of lattice constants, can hardly be obtained with the methods hitherto in general use. It occurred to us, however, that it should be possible to eliminate such sources of error (i) by suitable electromagnetic focusing and biasing of the beam, and (ii) by comparison of the diffraction pattern of the substance under examination with that produced by a standard structure, the constants of which had been previously determined with the utmost accuracy possible with X-rays. In preliminary experiments the two patterns were superimposed by interrupting the exposure and bringing the reference material into the electron beam by suitable manipulation of the specimen carrier, and then completing the exposure. It was found, however, that the identification of the rings was frequently difficult and in many cases rendered uncertain owing to the coincidence of rings common to both patterns. Increased scattering due to the double exposure also added materially to the difficulties of measurement. Accordingly it was decided to attack the problem on lines analogous to those followed in spectrography. The principle finally adopted consisted in exposing one half of the photographic plate to the diffracted rays emanating from the one specimen and then exposing the other half to electrons diffracted by the reference specimen, the electron-beam axis, concentration and velocity remaining unchanged.

§ 3. THE CAMERA

The electron-diffraction camera was constructed by the Cambridge Instrument Company Ltd., and is shown in section in figure 1. Although essentially similar in design to a smaller camera described elsewhere*, it incorporated, in addition to a new type of shutter, certain other features which added greatly to the convenience and speed of operation and which will now be described.

Cathode chamber assembly. The glass vessel A was made from a straight-necked wine bottle. After the base had been cut off, the neck and body apertures were ground normal to the axis with carborundum paste on a copper block. The vessel was then sealed with picein wax into a groove in the brass block B, the lower surface of which was ground flat to make a vacuum-tight joint with the similarly ground and suitably lubricated upper surface of the anode block. The flanges C and D protected the picein and grease from bombardment by the discharge and consequent gas evolution. The cathode assembly consisted of a highly polished massive aluminium rod E screwed into the brass block F, the lower surface of which, covered by a thin greased rubber washer, formed a vacuum-tight joint with the neck of the bottle. Since this joint was easily made and broken it was a simple matter to withdraw the cathode for repolishing and to remove sputtered metal from the walls of the discharge vessel. It was thus possible, without inconvenience, always to operate the discharge under the most favourable conditions. In the event of damage to the glass vessel, replacement with retention of symmetry about the axis was a simple matter, because the type of bottle used was moulded.

Electron-beam trap. Owing to the intensity of the electron beam obtained by focusing, the exposure times required were never greater than $\frac{1}{5}$ sec., and difficulty was sometimes experienced in giving a sufficiently short exposure by hand operation of the shutter. An electron-beam trap was therefore incorporated in the instrument, enabling the exposures to be made and timed electrically. The trap, figure 1, consisted essentially of a trapping chamber G with a wide diaphragm situated in the axis of the main tube. By applying suitable potentials to the trapping plates P_1 and P_2, the electron beam could be caused to pass through the diaphragm or could be collected in the trapping chamber. In the case of reflection experiments the polarity of the trapping plates with the beam trapped was such that the beam was withdrawn behind the specimen.

Focusing-coil assembly. A bank of four coils, as shown in figure 1, electrically connected as two independent pairs, was adjustable into any position between the diffraction chamber and the main evacuation port. The axis of the coil could be inclined to that of the camera by means of the levelling screws S, enabling the electron-beam axis to be biased into any desired direction.

Specimen-carrier assembly. In the case of reflection experiments the specimen under examination was clamped side by side with the reference specimen on the plate P, provided with levelling screws by means of which the surfaces of the two specimens could be adjusted into a common plane perpendicular to the axis of

* *Loc. cit.*

rotation of the stem. The plate could be unscrewed from the stem and replaced by a similar plate or by a holder suitable for transmission experiments. Controls, the nature of which will be evident from figure 1, enabled the specimen to be adjusted as follows: (i) inclined to the electron beam, (ii) translated into the beam, (iii) rotated in the azimuthal plane and, finally, (iv) moved laterally in the specimen plane in a direction at right-angles to the beam.

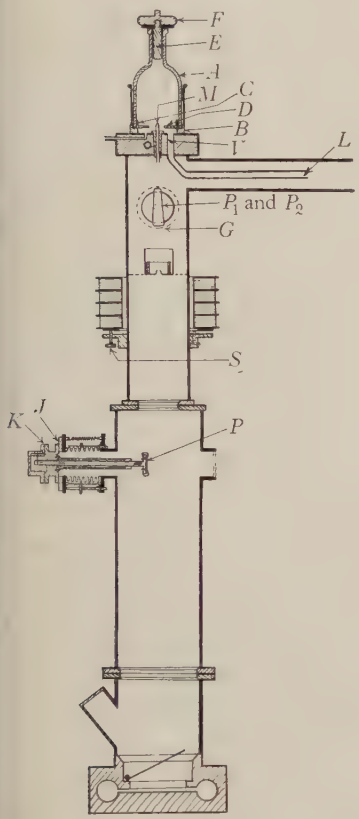


Figure 1. The electron-diffraction camera.

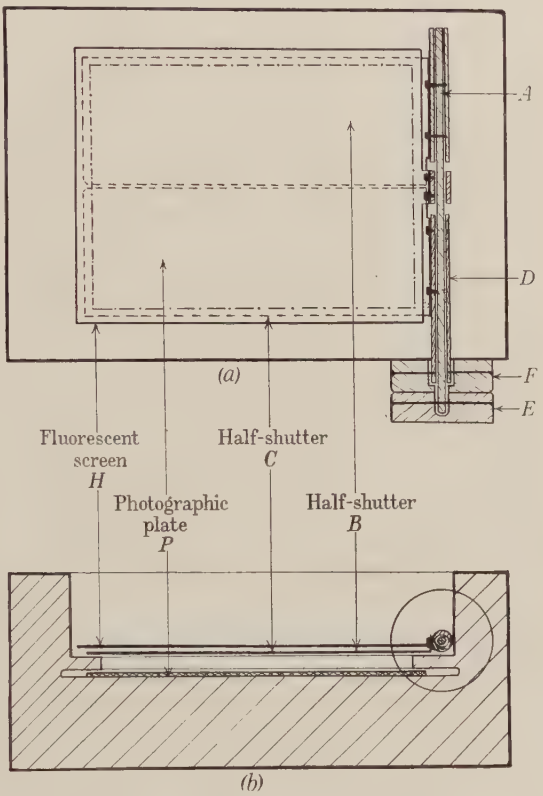


Figure 2. Details of camera shutter.

The necessary flexibility combined with vacuum-tightness in the assembly was obtained by means of two lapped surfaces at J and K and a short length of flexible metallic bellows*.

Camera shutter. The construction is shown in cross-section and elevation in figures 2 (*a*) and 2 (*b*) respectively. The fluorescent screen H was hinged to rotate freely about the stem A, to which the half-shutter B was secured. The other half-shutter C was similarly screwed to the tube D, concentric with A. Rotation through 90° of either head E or F raised the half-shutter B or C respectively, together with the loosely hinged fluorescent screen, thus exposing the corresponding half of the

* Replacement of the specimen-carrier section by a tube section fitted with crossed deflecting-plate pairs converted the instrument into a high-speed cathode-ray oscillograph.

photographic plate. The heads E and F were suitably ground and lubricated to make vacuum-tight joints. When required the two shutter halves could be raised and lowered together.

§ 4. THE VACUUM TECHNIQUE

The degree of vacuum required in the cathode chamber for the production of an electron beam of suitable velocity is of the order of 10^{-3} mm. Schobitz* has shown that in order to avoid loss of definition in the diffraction patterns the pressure in the diffraction chamber must be considerably less than 10^{-5} mm. In the instrument described above these conditions were fulfilled by means of the differential pumping system shown in figure 1. Whilst the 4-stage mercury-vapour pump was directly connected by a wide-bore (5-cm.) tube to the main body of the camera, the cathode chamber was evacuated through a 1·2-cm. bore tube L, the effective orifice of which could be reduced by means of the valve V. Since a steady flow of gas into the cathode chamber was maintained by means of the capillary leak system described elsewhere †, the conditions obtaining in the vicinity of the main evacuation port were similar to those of an injector. The arrangement proved so satisfactory, not only in the attainment of the necessary vacua but also in preventing diffusion of the mercury vapour into the camera, that it was found possible to dispense with a mercury-vapour trap.

The vacuum-tightness of the instrument depended upon that of thirteen flat lapped-surface joints ranging in external diameter between 4 and 15 cm. By experience it was found that the optimum width of lapped surfaces was about 2 cm., the surfaces being ground with coarse-grade carborundum paste on a steel surface plate. Whereas a more finely finished surface failed to hold the grease whilst under vacuum, the matt surface obtained by the above method prevented flow of grease. The grease was made by digesting 2 per cent of pure crêpe rubber in pure white petroleum jelly in vacuo in a boiling brine bath. Semi-permanent joints such as observation windows etc. were made with picein wax.

§ 5. PRODUCTION OF THE ELECTRON BEAM

For the purpose of accurate electron-diffraction analysis it is essential that the beam velocity and intensity be uniform during exposure. These requirements cannot be fulfilled by means of a hot cathode, because this would entail the use of a constant-voltage source of high tension. On the other hand, a cold-cathode discharge tube fed through a suitable current-controlling device offers a means for the production of a beam of steady current and voltage with a minimum outlay in smoothing equipment. Accordingly we have incorporated in our instrument a cold cathode fed through a diode operated under conditions of saturation. The high-tension condenser was filled to a peak voltage of 110 kV. by half-wave rectification of a 50 ~ transformer output. The maximum beam voltage employed in our experiments was 70 kV. The capacity of the condenser was such that with a discharge-chamber

* *Phys. Z.* **32**, 37 (1931).
† *Loc. cit.*

current of 1 mA. the valve was still saturated throughout each idle half-cycle, the storage voltage never falling below about 90 kV. With a constant current flowing through the cold-cathode discharge chamber the beam velocity depended solely upon the gas pressure, which could be adjusted and maintained uniform over long periods by means of the capillary leak system.

§ 6. THE FOCUSING OF THE ELECTRON BEAM

The electron beam was focused in order to increase the ratio of beam to cathode-chamber current and the resolving power, and also to enable the beam to be separated from molecular rays. The first of these objects was attained by electrostatic focusing brought about by suitable design of the cathode chamber and diaphragm. In connection with this it is important to note that the diaphragm tube *M*, figure 1, protruded well into the cathode chamber and was shaped in such a manner as to subject the beam to an intense concentrating field. Whilst the bore of the diaphragm tube was 5 mm., its cathode-chamber end was spun down to an aperture of 0·1 mm. The beam issuing from the diaphragm tube was divergent, but could be rendered suitably convergent by means of the focusing coils. The following considerations lead to the conclusion, confirmed in practice, that the best results are to be obtained when the undiffracted beam is brought to a focus on or very slightly below the photographic plate. To fix ideas, consider a transmission specimen consisting of a film of randomly disposed crystals. Since the angle between incident and emergent paths of electrons of given velocity is constant for any set of Bragg planes, it is easy to see by a simple geometrical construction that a divergent beam will continue to diverge after diffraction, a parallel beam will remain parallel and, finally, a beam converging to a focus in the vicinity of, but rather below, the plate will result in the diffraction rings attaining maximum definition on a curved surface which is, with an effective camera-length of 50 cm., to all practical purposes coincident with the plate.

By means of levelling-screws the focusing coils could be tilted and the electron beam thus separated from the molecular rays, which could then be trapped and thus prevented from striking either specimen or photographic plate. Furthermore, by progressive tilting of the focusing coils the specimen could be rapidly and systematically explored.

§ 7. PROCEDURE

Two specimens, one for examination and the other for reference, were mounted side by side on the specimen-carrier, which was adjusted in such a manner that, when one specimen was in a suitable position in the electron beam, either simple rotation through 180° or a short predetermined horizontal traverse of the specimen-carrier would bring the second specimen into a similar position. The first half of the plate was then exposed, whereupon the second specimen was brought into the beam and the exposure completed on the second half of the plate. The whole operation could be completed in about 1 second. All conditions affecting the production of the electron beam were unchanged throughout these operations.

The reference specimen. The choice of the substance serving for the preparation of the reference specimen was governed by the following requirements. The specimen should (i) be easy to prepare in a reproducible manner for use in transmission or reflection; (ii) yield a large number of well-defined sharp rings, each of uniform intensity and without too pronounced a contrast of intensity between the rings; and finally (iii) consist of a substance the lattice constants of which have already been determined with great accuracy by X-ray diffraction. Specimens consisting of Pt, Au, Ag, Al, Mg, Zn, Cu, Fe, SnO_2, PbO_2, Al_2O_3 and ZnO respectively were examined for their suitability as reference materials. Of these, zinc oxide alone, prepared by the combustion of zinc vapour, fulfilled the necessary requirements. For the purposes of reflection the zinc-oxide smoke was deposited on a polished metal disc, whilst for transmission the oxide was collected on a fine-mesh nickel gauze. A number of such specimens were examined and the resulting diffraction patterns, of which that shown in figure 3 is characteristic, were so uniform as to establish clearly the reproducibility of the crystalline structure of specimens obtained in this manner. Six transmission and three reflection patterns were completely analysed. All planes within the range covered by the patterns were accounted for, and no forbidden rings occurred. Within the limits of experimental error inherent in previous methods of electron diffraction the values of the lattice constants obtained in each case agreed with the values given by Bragg* ($a = 3{\cdot}220$ Å.; $c = 1{\cdot}608$). In what follows all the measurements have been referred to these values, and in the event of their revision being found necessary it will be a simple matter to apply the appropriate correction to the results set forth below.

The preparation of the specimens of zinc oxide for examination, as distinct from those serving for reference purposes, will be described below.

§ 8. ANALYSIS OF THE DIFFRACTION PATTERNS

Analysis was carried out as follows. The reciprocals of the radii of the reference-pattern rings were plotted on a logarithmic scale and fitted to the suitable Hull and Davey† nomogram. In addition to identifying the rings in terms of the corresponding Bragg planes, this procedure also served to check the structure of the reference material. Spacings calculated from the accepted X-ray values were then assigned to the reference rings. Thereupon the rings due to the specimen under examination were measured by interpolation between adjacent reference rings and the corresponding Bragg-plane spacings calculated. The further procedure was as previously outlined.

It will be realized that the use of a reference pattern eliminates errors due to inaccuracy in measurement of accelerating voltage and effective length of the camera, together with those due to the photographic plate being flat instead of spherical about the point of diffraction. Distortion due to shrinkage in the emulsion of the plate and any lack of uniformity in the concentrating field in the planes at right angles to the undiffracted beam do not affect the measurements. Moreover it is

* *Phil. Mag.* **39**, 647 (1920).

† *Phys. Rev.* **17**, 549 (1921).

not necessary to assume the validity of de Broglie's law, but merely that a definite value of wave-length is to be associated with electrons of a given velocity. Since in all our experiments the beam was biased well away, usually about 5 cm., from its unfocused position, the occurrence of any change in the conditions associated with the production of the beam itself was betrayed by a displacement of one set of rings relative to the other. Figure 4 shows the effect produced by a small deliberate voltage-change between the two half-exposures. It will be seen that corresponding half-rings differ in diameter and are not described about a common centre. Furthermore, owing to its intensity at the photographic plate the beam of undiffracted electrons in many cases actually drilled a minute hole in the emulsion. Thus it was generally possible to tell at a glance whether a given plate should be retained for analysis or discarded.

§ 9. EXPERIMENTAL RESULTS AND DISCUSSION

In what follows, electron-diffraction patterns due solely to specimens of condensed zinc oxide crystallizing in a triangular close-packed lattice in the hexagonal system, with a basal axis of 3·220 Å. and an axial ratio of 1·608 and in which the crystal array was wholly random, are termed *reference patterns*, all others being referred to as *specimen patterns*. All patterns were obtained by transmission, the general plane of the diffracting film being normal, or nearly so, to the electron beam, unless otherwise stated.

The specimens were prepared by withdrawing a nickel wire loop 1 cm. in diameter from a skimmed melt of granulated zinc of forensic quality in a quartz crucible. A double-shutter photograph of the transmission patterns given by such a specimen and a normal zinc-oxide reference film is reproduced in figures 5 (A) and 5 (B), being the reference and specimen patterns respectively. In spite of the remarkable dissimilarity between these two patterns, the coincidence of certain rings in both is so exact as to enable the conclusion to be drawn that the specimen consisted, at least in part, of zinc oxide exhibiting the normal structure. The complete analysis of figures 5 (A) and 5 (B) is given below in table 1.

Thus, according to the analysis set forth in table 1, the specimen consisted of zinc oxide of normal crystal structure and lattice constants, together with zinc and zinc oxide basally pseudomorphic therewith. The relatively feeble intensities of the rings due to zinc and the pseudomorphic oxide strongly suggest that these substances were present in comparatively minute proportions, the specimens consisting for the most part of normal zinc oxide. Apart from the presence of rings due to such traces of zinc and pseudomorphic zinc oxide in the specimen pattern, the outstanding difference between figures 5 (A) and 5 (B) lies in the totally different distribution of intensities between the rings, and it was mainly owing to this disparity that W. L. Bragg and J. A. Darbyshire*, on obtaining electron-diffraction photographs essentially similar to that reproduced in figure 5 (B), assigned a face-centred cubic structure to films prepared in the manner outlined above.

* *Trans. Farad. Soc.* **28**, 522 (1932).

Table 1

Zinc oxide reference		Partially oxidized zinc specimen			
			Origin attributed to planes		
Plane	Spacing d/N (Å.)	Spacing (Å.) obtained by interpolation	Normal ZnO	ZnO basally pseudomorphic with Zn	Zinc
—	—	3·79	(1.0.2)½	—	—
(1.0.0)	2·789	2·789	(1.0.0)	—	—
(0.0.1)2	2·589	—	—	—	—
(1.0.1)	2·456	2·456	(1.0.1)	—	—
—	—	2·327	—	(1.0.0)*	(1.0.0)*
—	—	2·116	—	—	(1.0.1)*
(1.0.2)	1·897	1·897	(1.0.2)	—	—
—	—	1·722	—	—	(1.0.2)†
(1.1.0)	1·610	1·610	(1.1.0)	—	—
(1.0.3)	1·468	1·468	(1.0.3)	—	—
(1.0.0)2	1·395	1·395	(1.0.0)2	—	—
(1.1.2)	1·368	—	—	—	—
(2.0.1)	1·347	1·347	(2.0.1)	(1.1.0)†	(1.1.0)†
(0.0.1)4	1·295	—	—	—	—
(1.0.1)2	1·228	1·228	(1.0.1)2	—	—
(1.0.4)	1·174	—	—	—	—
—	—	1·163	—	(1.0.0)2†	(1.0.0)2†
(2.0.3)	1·085	—	—	—	—
(1.2.0)	1·054	1·054	(1.2.0)	(1.1.4)	(1.0.1)2
(1.2.1)	1·033	1·033	(1.2.1)	—	—
(1.1.4)	1·009	—	—	—	—
(1.2.2)‡	0·976‡	0·976‡	(1.2.2)‡	—	—
(1.0.5)	0·971	0·971	(1.0.5)	—	—
(1.0.2)2	0·949	—	—	—	—
(1.0.0)3	0·930	0·930	(1.0.0)3	—	—
(1.2.3)	0·900	0·900	(1.2.3)	—	—
(3.0.2)	0·875	0·875	(3.0.2)	—	—
(1.1.0)2	0·805	0·805	(1.1.0)2	—	—

The lattice constants of the components of the partially oxidized zinc specimen, obtained by reference to normal zinc oxide of assumed constants, $a = 3{\cdot}220$ Å.; $c = 1{\cdot}608$, were as follows:

Pseudomorphic zinc oxide $a = 2{\cdot}686 \pm 0{\cdot}002$ Å.; $c = 2{\cdot}56$.

Normal zinc $a = 2{\cdot}686 \pm 0{\cdot}002$ Å.; $c = 1{\cdot}86$.

The origin of the faint innermost ring will be discussed below.

* Denotes a spot pattern superimposed on a ring.

† Denotes one or more spots due to the same set of planes.

‡ Signifies not resolved.

Now, it does not seem to have been realized hitherto that crystal orientation can under certain circumstances give rise to an abnormal distribution of electron-diffraction ring intensities without any splitting up of the rings into arcs. Owing to deep penetration, the intensity distribution between the rings in X-ray diffraction powder photographs is of great assistance in their analysis because the ring intensities may, as a rule, be taken as being proportional to the atom-densities in

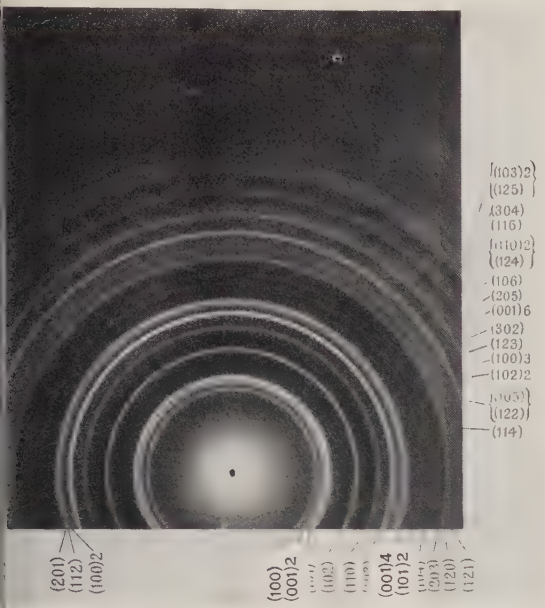


Figure 3. Normal zinc oxide.

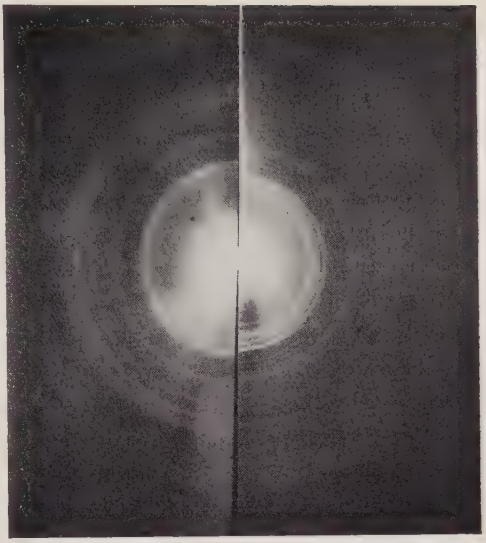
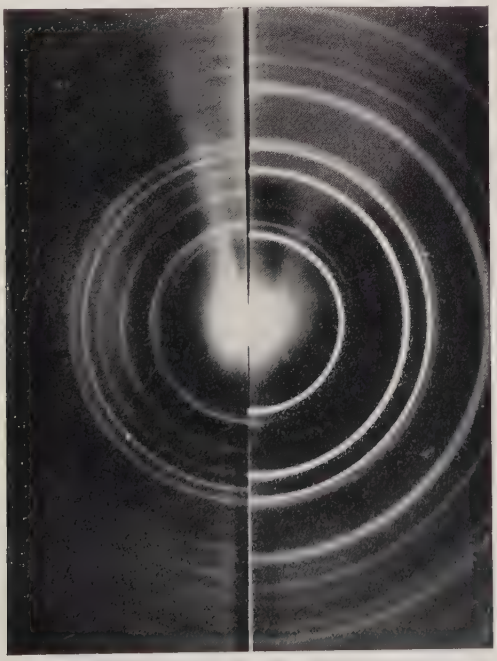


Figure 4. Faulty registration due to change in voltage.



A Reference pattern

B Specimen pattern

Figure 5.

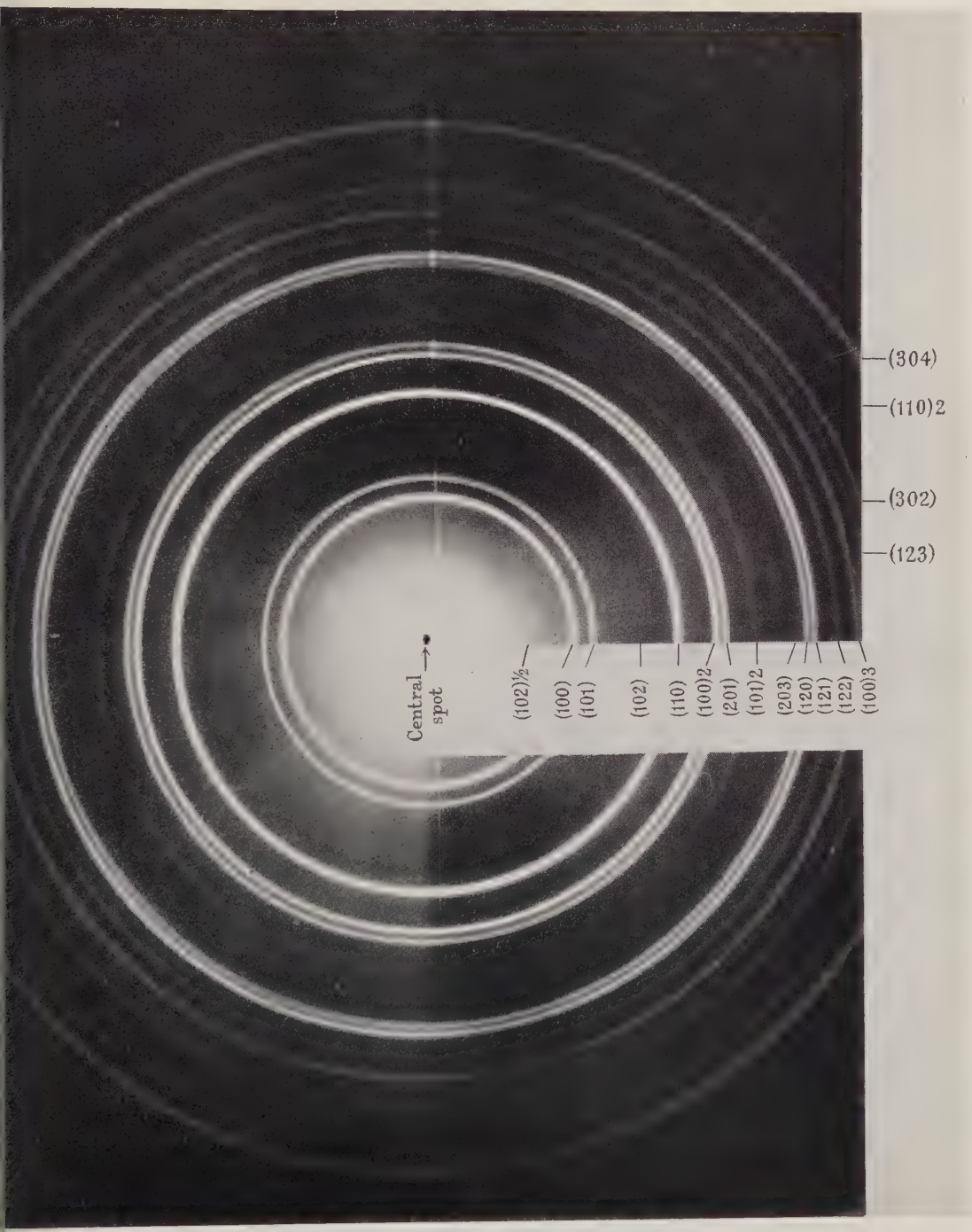


Figure 6. Normal zinc oxide formed by complete oxidation of a zinc film.

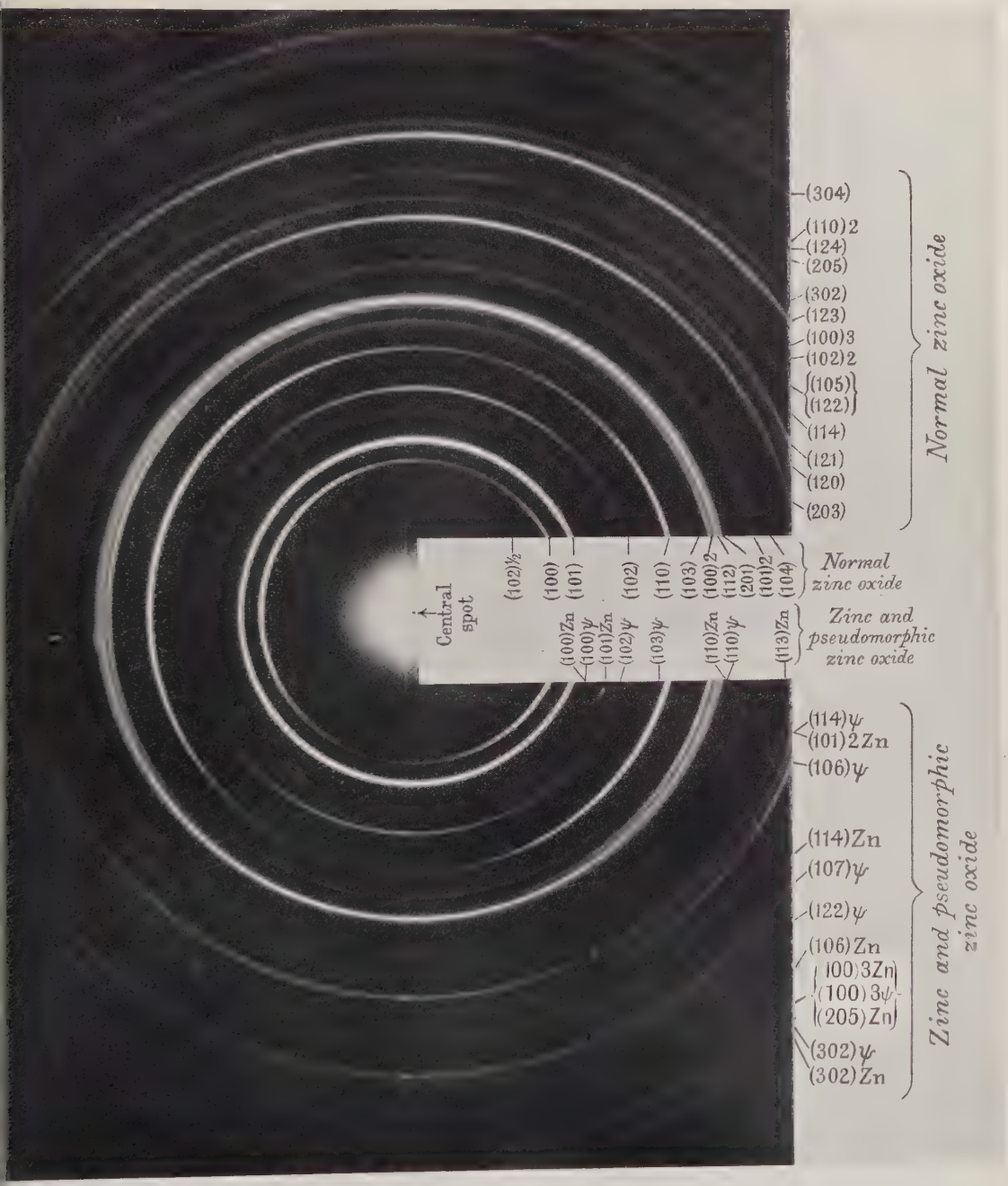
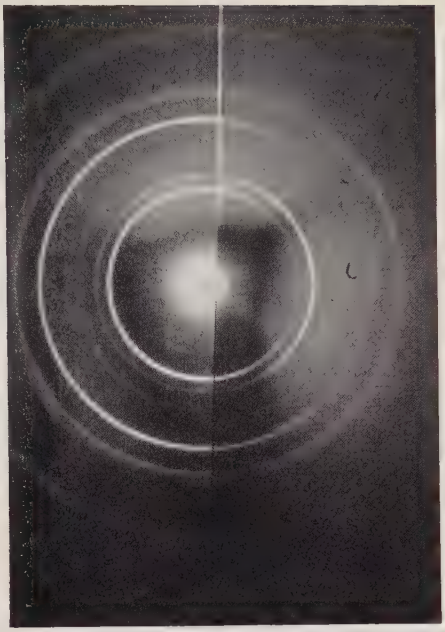

Central spot
(102)½
(100)
(101)
(102)
(110)
(103)
(100)2
(112)
(201)
(101)2
(104)
Normal zinc oxide
(100)Zn
(100)ψ
(101)Zn
(102)ψ
(103)ψ
(110)Zn
(110)ψ
(113)Zn
Zinc and pseudomorphic zinc oxide
(304)
(110)2
(124)
(205)
(302)
(123)
(100)3
(102)2
(105)
(122)
(114)
(121)
(120)
(203)
Normal zinc oxide
(114)ψ
(101)2Zn
(106)ψ
(114)Zn
(107)ψ
(122)ψ
(106)Zn
(100)3Zn
(100)3ψ
(205)Zn
(302)ψ
(302)Zn
Zinc and pseudomorphic zinc oxide

Figure 7.

Figure 8.



A *B*

Figure 10.

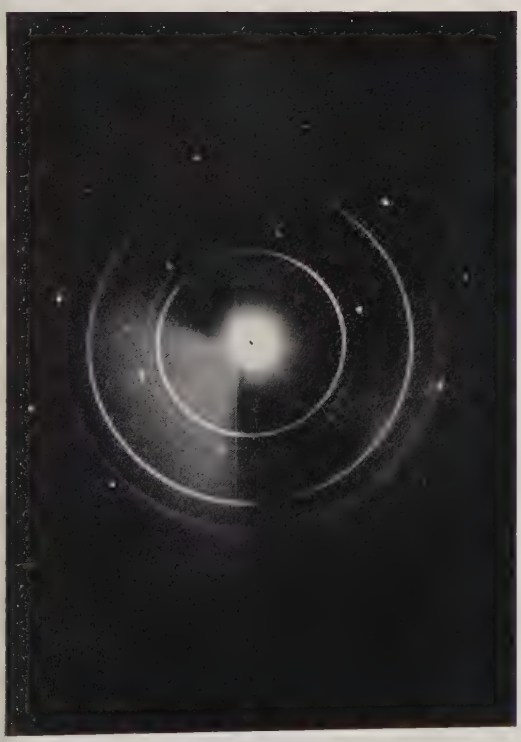


Figure 11.

Figure 12.

the corresponding planes, and it seems to have been generally supposed that this practice can also be extended to electron-diffraction patterns provided that the rings are entire, i.e. of uniform intensity throughout each individual ring. Owing to the limited penetrating powers of the medium-speed (20 to 70-kV.) electrons usually employed, however, this is not permissible unless, indeed, it be already known that the crystals are randomly disposed. Consider, for example, a plane diffracting film consisting, to take the most symmetrical case, of cubic crystals orientated with two axes X, Y parallel to the film, but otherwise without constraint. Under these conditions reflection patterns will consist solely of spots forming a complete pattern indistinguishable from that given by a single crystal where it is rotated about a Z-axis maintained perpendicular to the plane of the film and therefore also normal to the electron beam. On the other hand, the corresponding transmission pattern obtained with the beam normal to the film will consist of complete rings of uniform intensity throughout each ring. The relative intensities of these rings, however, will differ from that produced by an array of randomly disposed crystals, in that only rings of zero third index will appear. Thus, the patterns will correspond to that given by a single crystal rotating about an axis coinciding with that of the electron beam. With gradual departure from an orientation perfect in the sense postulated, the reflection pattern will degenerate from spots into arcs, whilst hitherto absent rings will appear in the transmission pattern. In general, a plane polycrystalline film in which the directions of the crystals are random may be regarded in electron diffraction as equivalent to a single crystal exercising to the full both rotational degrees of freedom. All possible planes will be represented in the diffraction pattern, the relative ring intensities being proportional in the main to the atom-densities in the corresponding planes. Similarly, a film in which the crystals are orientated in a common direction may be treated as a single crystal with only one degree of rotational freedom and, if the axis of rotation coincide with that of the electron beam, the diffraction pattern will consist of complete rings formed solely by plane systems parallel, or nearly so, to the beam axis. Inclination of the rotational axis to the beam must result in the formation of a pure spot pattern. Finally, with the extreme degree of orientation we have, in fact, a single crystal devoid of both degrees of rotational freedom, and no matter what the directions of restraint may be a pure spot pattern will result. Thus, in electron-diffraction a spot pattern need not necessarily be due to a single-crystal structure, nor does a pattern of complete rings imply absence of orientation. Further, arcing of the rings, far from being a criterion of orientation, is rather to be regarded as evidence of imperfect orientation. Similar considerations apply to cases of lesser symmetry such as, for example, the hexagonal system.

In the light of these considerations it is now possible to put forward a reasonable explanation of the abnormal distribution of the ring intensities shown in figure 5 (B), and also in the patterns, figures 6, 7 and 8, obtained from other specimens prepared under approximately similar conditions. The normal zinc-oxide rings identified in these patterns, together with those of figures 5 (A) and (B), are classified in table 2 in order of decreasing steepness of their parent planes, together with the estimated

relative ring intensities. Also, where rings show a tendency to split up into arcs the approximate directions of arcing relative to that exhibited by the (1.0.0) ring are likewise recorded in table 2.

Table 2

Plane	Angle with basal planes (degrees)	Figure 5 A	Figure 5 B	Figure 6	Figure 7		Figure 8	
		Reference intensity*	Intensity	Intensity	Intensity	Arcs†	Intensity	Arcs
(1.0.0)	90	M	SS	SS	S	→	S	—
(1.1.0)	90	FM	SS	SS	MS	→	MS	—
(1.0.0)2	90	F	S	S	FM	→	FM	—
(1.2.0)	90	VF	M	M	VF	→	VF	—
(1.0.0)3	90	VF	M	F	VVF	→	VF	—
(1.1.0)2	90	VF	F	F	VVF	→	Outside	range
(2.0.1)	75	F	M	M	FM	→	FM	—
(1.2.1)	72	VF	M	M	VF	→	F	—
(1.0.1)	62	MS	MS	MS	SS	↑	SS	↑
(1.0.1)2	62	VF	Nil	VF	F	↑	VF	↑
(3.0.2)	60	VF	VVF	VF	VF	→	VF	—
(1.2.2)‡	57	F	VF	VF	F	→	VVF	—
(2.0.3)	51	F	Nil	Nil	F	↑	F	↑
(1.2.3)	45	F	VF	VVF	F	↑	F	↑
(1.0.2)	43	M	VF	Nil	M	↑	M	↑
(2.0.5)	37	VF	Nil	Nil	VVF	?	Outside	range
(1.0.3)	32	M	VF	Nil	F	↑	VF	↑
(1.1.2)	30	M	Nil	Nil	M	↑	M	↑
(1.0.4)	25	VF	Nil	Nil	VVF	?	Nil	
(1.0.5)‡	20	F	VF	VF	F	→	VVF	—
(1.0.6)	17	VF	Nil	Nil	Nil		Outside	range
(1.1.4)	16	VF	Nil	Nil	VVF	↑	VVF	↑
(0.0.1)2	0	M	Nil	Nil	Nil		Nil	
(0.0.1)4	0	VF	Nil	Nil	Nil		Nil	
(0.0.1)6	0	VVF	Nil	Nil	Nil		Nil	

* The ring intensities are graded in order of increasing intensity as follows:
Nil—VVF—VF—F—FM—M—MS—S—SS.

† The arrows indicate the general directions of maximum arcing intensity.
‡ These two rings were unresolved.

All rings in figure 6 are due solely to normal zinc oxide, and their relative intensities are similar to those of the normal zinc-oxide rings in figure 5 (B). Thus, in putting forward an explanation of their relative ring intensities both patterns, figures 5 (B) and 6, can be considered together.

In the first place, rings due to planes of zero third index are, without exception, much more intense than the corresponding rings in the reference pattern, figure 5 (A), whilst rings of (0.0.z) type are completely absent, although in the case of the reference film the (0.0.1)2 planes give rise to one of the most prominent rings in the normal pattern, figure 5 (A). In view of these facts it is difficult to resist the conclusion that, in the case of figures 5 (B) and 6, the specimen contained practically no crystals with the *Z*-axis in the plane of the film, i.e. normal to the electron beam,

and that a large proportion of the crystals were oriented in such a manner that their *Z*-axes were perpendicular to the film and thus parallel to the beam axis. Furthermore, though rings due to the (2.0.1) and (1.2.1) planes are likewise strengthened, the (1.0.1) rings in figures 5 (B) and 6 are approximately equal in intensity to the corresponding ring in figure 5 (A), whilst with still further decrease in plane steepness the specimen rings become progressively weaker than their reference prototypes. Thus it would appear that practically all the crystals in the specimens producing the patterns in figures 5 (B) and 6 were orientated between limits such that the inclination of the basal planes to the beam axis varied between about 60° and 120°, the great bulk of the crystals, however, being so orientated that this angle was 90°. These views are summarized in graphical form in figure 9, curve I. With

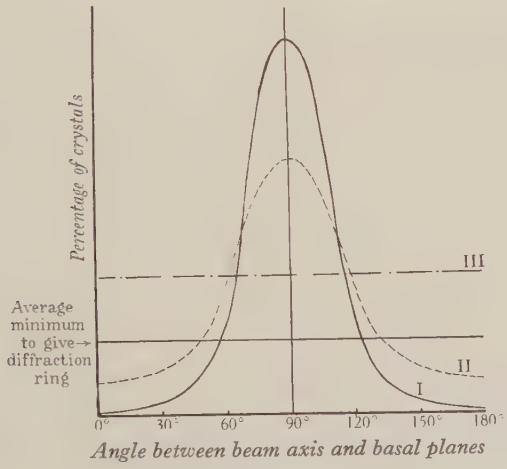


Figure 9.

decreasing extent of orientation the curve flattens out as shown in curve II until, with the random element complete, as in the zinc-oxide reference, the distribution is given by the straight line III. It will be appreciated that as the degree of orientation becomes less, diffraction rings will be observed from planes previously not in a position to diffract sufficiently to affect the photographic plate.

The patterns in figures 7 and 8 were obtained as previously with the general plane of the film normal to the electron beam. Nevertheless, they differ from figures 5 (B) and 6 as follows: (i) the rings show a tendency to break up into arcs; (ii) several rings missing from figures 5 (B) and 6 now appear; (iii) the zone of maximum increase in intensity as between reference rings, figure 5 (A), and specimen pattern rings, figures 7 and 8, has shifted from the region of the plane of zero third index to that of the (1.0.1) planes; and finally (iv) faint spot patterns are visible.

The pattern-changes (i), (ii) and (iii) suggest that whilst the general plane of the specimen was normal to the electron beam, either part or the whole of the film area actually examined was itself inclined to the electron beam at an angle of rather

less than 30°. This view is strongly supported by the observed directions of arcing recorded in table 2 and was confirmed experimentally by the double shutter record, figures 10 (A) and (B). In obtaining these patterns a half-shutter exposure, figure 10 (A), was first made with the specimen plane normal to the beam. The second half-shutter record, figure 10 (B), was obtained after the specimen had been rotated through an angle of approximately 30° about an axis lying in the plane of the film and passing through the point of impact of the beam axis with the specimen. The resemblance between figures 10 (A), 5 (B) and 6, and between figures 10 (B), 7 and 8, is such as to establish clearly the views set forth above.

The diameter of the innermost and usually extremely faint ring visible in all the specimen patterns excepting figure 8 corresponds to a Bragg plane spacing of 3·79 Å. Although the directions of arcing of this and of the (1.0.2) normal zinc-oxide rings are different, we can at the moment only suggest that the innermost ring is due to the half-order reflection from the (1.0.2) planes of normal zinc oxide, the spacing of which is 1·897 Å. It is significant that the intensity of this suggested half-order ring increases with increasing orientation of the normal zinc-oxide crystal bases in the plane of the specimen film.

The faint traces of spot patterns visible in figures 5 (B), 7, 8 and 10 proved on analysis to be due to both zinc and the pseudomorphic oxide. Further experiments were carried out with the object of obtaining specimens exhibiting more pronounced spot patterns. Such specimens were successfully prepared by slow cooling of films after withdrawal from the melt. To illustrate the results obtained it will suffice to reproduce the patterns, figures 11 and 12, obtained from one such specimen. The rings in both patterns are due to the normal form of zinc oxide. The spot patterns, however, are entirely due to zinc and zinc oxide basally pseudomorphic therewith.

The spot patterns in figure 11 must be due to a single crystal because all possible spots of the planes having zero third index are present, whilst no diffraction occurs from planes of (0.0.z) type; this proves that the Z and electron-beam axes coincided. Owing to basal-plane pseudomorphism, spots due to planes of zero third index are common to both zinc and the pseudomorphic oxide. The specimen contained, however, other crystal fragments which were in a position to produce rings from such steep planes as the (1.0.1) and (1.2.1) of zinc and the (2.0.1) of pseudomorphic zinc oxide. In the case of figure 12, the specimen plane had been rotated to make an angle of about 80° with the electron beam, with the result that some of the spots due to planes of zero third index have disappeared and others produced by less steeply inclined planes now appear. These results are set forth in detail in table 3.

A close scrutiny of the above and many similar diffraction patterns has failed to reveal any signs whatever of the existence of a zinc oxide with a structure intermediate between the normal and pseudomorphic forms. Nevertheless, it does not seem reasonable to suppose that the transition between the two forms occurs per saltum. It is much more probable that the change occurs in an exponential manner such that the amount of zinc oxide in the transition stage is so small as to have escaped detection.

Table 3. Analysis of the spot patterns in figures 11 and 12. Arcs and complete rings are due to normal zinc oxide and are not included in the analysis

Spacing d/N (Å.)	Figure 11		Figure 12	
	Zinc	Pseudomorphic zinc oxide	Zinc	Pseudomorphic zinc oxide
2·326	(1.0.0)	(1.0.0)	(1.0.0)	(1.0.0)
2·204	—	—	—	(1.0.1)
2·116	(1.0.1)	—	(1.0.1)	—
1·722	—	—	(1.0.2)	—
1·343	(1.1.0)	(1.1.0)	(1.1.0)	(1.1.0)
1·163	(1.0.0)2	(1.0.0)2	(1.0.0)2	(1.0.0)2
1·147	—	(2.0.1)	—	(2.0.1)
1·133	(2.0.1)	—	(2.0.1)	—
1·102	—	—	—	(1.0.1)2
0·907	(1.2.1)	—	—	—
0·879	(1.2.0)	(1.2.0)	(1.2.0)	(1.2.0)
0·872	—	—	—	(1.2.1)
0·861	—	—	(1.0.2)2	—
0·852	—	—	—	(1.2.2)
0·775	(1.0.0)3	(1.0.0)3	(1.0.0)3	(1.0.0)3

§ 10. CONCLUSION

In all, 42 patterns obtained from oxidized-zinc films and surfaces have now been examined in the course of the investigations outlined above and in a previous communication to the Royal Society*, with the result that certain facts have been established which suggest an explanation of the well-known corrosion-resisting properties of zinc. These facts may be summarized as follows: (i) When a zinc film is oxidized at room temperature in air, only the pseudomorphic modification of zinc oxide is formed; (ii) such pseudomorphic zinc-oxide films are sufficiently thin to be partially transparent to 40-kV. electrons; (iii) heating converts the pseudomorphic oxide into zinc oxide of normal structure which, however, is still separated from the metallic zinc by a freshly formed intermediate layer of pseudomorphic zinc oxide; (iv) when zinc appears in the form of a single crystal the pseudomorphic oxide does so likewise; (v) single crystal structure of the normal form of zinc oxide has so far not been observed in our experiments; and finally (vi) in the case of single-crystal films the hexagonal face invariably lies parallel, or nearly so, to the general plane of the film, and even in the polycrystalline films there is a pronounced orientation of a similar nature.

From its position in the periodic system zinc is a metal which might be expected to be peculiarly prone to atmospheric attack; but, as we have now shown experimentally, a fresh zinc film on exposure to air at room temperatures becomes covered with an extremely thin film of crystalline zinc oxide basally pseudomorphic with the zinc. The forces giving rise to the phenomenon of basal-plane pseudomorphism in this case also suffice to cause the pseudomorphic oxide to retain any

* *Loc. cit.*

single-crystal structure originally present in the case of the zinc film. The pseudomorphic zinc-oxide crystals in effect may be regarded as continuations of the zinc crystals, growing out of their hexagonal faces. These hexagonal faces tend to be orientated in the surface of the film. The atoms in the pseudomorphic zinc-oxide lattice are abnormally closely packed and thus difficult of penetration by oxygen, and the impenetrability by oxygen of the structure as a whole is assured by the pseudomorphic properties of the pseudomorphic zinc oxide and by the tendency of the zinc crystals to orientate themselves in such a manner that their hexagonal faces lie in the surface, conditions which all tend to ensure the formation of a completely continuous and, in the event of injury, self-healing film of pseudomorphic zinc oxide.

When zinc is oxidized at an elevated temperature the pseudomorphic oxide, which is stable at ordinary temperatures, is formed first and subsequently changed into the normal form of zinc oxide. Furthermore, since single-crystal films of the normal zinc oxide have so far not been observed by us, although such films of zinc and pseudomorphic zinc oxide were frequently obtained, it would appear that a single-crystal layer of pseudomorphic zinc oxide, in forming the normal oxide under the action of heat, disintegrates from a single crystal into a polycrystalline layer, owing to the considerable increase in basal dimensions accompanying the change from one modification of the oxide to the other.

§ 11. ACKNOWLEDGMENTS

We wish to thank the Cambridge Instrument Company, Ltd. for the gift of the electron-diffraction camera and the loan of a travelling microscope, Messrs Ferranti, Ltd. and Imperial Chemical Industries, Ltd. for grants and apparatus, and the Department of Scientific and Industrial Research for a grant which enabled one of us (A. G. Q.) to devote his whole time to this research.

DISCUSSION

Dr G. F. New. The results of this investigation are of interest in connection with the adhesion of oil paints to zinc surfaces. It is now apparent that the adhesion is in the first place to an oxide film instead of metal and this may explain the difficulties of producing satisfactory practical results. Although ordinary zinc oxide reacts with some types of linseed oil very strongly, it is possible that the pseudomorphic film of oxide is less reactive.

535.338.1:546.14.02

THE SPECTRA OF Br v, Br vi and Br vii

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ABSTRACT. The vacuum spark spectra of bromine have been investigated under different degrees of excitation in the region λ 1400 to λ 400, by means of a Siegbahn spectrograph. From a careful scrutiny of the plates the lines have been assigned to the different stages of ionization of the element. With the aid of these the principal members of the spectra of Br v, vi and vii, involving the low-lying terms, have been identified.

§ 1. INTRODUCTION

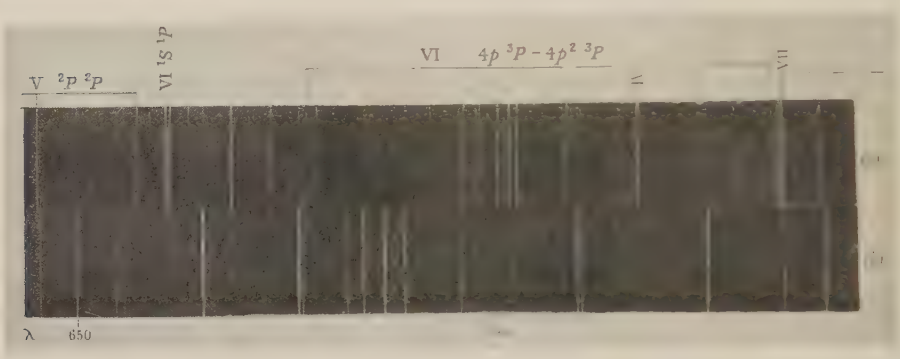
The experimental observations that have hitherto been made on the spectrum of bromine in the extreme ultra-violet consist chiefly of those by L. and E. Bloch[1] who photographed the region between λ 6000 and λ 2200, using a high frequency electrodeless discharge through bromine vapour as their source. But the spectrum does not appear to have been investigated for series relationships below λ 1200, in which region many of the principal groups of the higher spark spectra of the element are expected to occur. With the exception of a few pairs of Br ix, we have no knowledge of the characteristics of these higher spark spectra. It is the purpose of the present paper to give an account of the regularities that have been discovered in the spectra of Br v, Br vi and Br vii, based mainly on a careful experimental study of the lines under different degrees of excitation in the vacuum grating region. An extensive use is made of the regular and irregular doublet laws in the location and detection of the chief multiplets in the above spectra.

§ 2. EXPERIMENTAL

The experimental work which formed the essential basis of the present paper was carried out by K. R. Rao in Prof. Siegbahn's laboratory at Upsala. The source used was a highly condensed vacuum spark between electrodes of Al tipped with the bromides of Cs and Rb. The intensity of the spark was altered by varying the capacity of the condenser from 0·27 μF. to 0·09 μF. and also by introducing an inductance of about 0·5 mH. into the secondary circuit. Photographs of the spectra of two different bromides helped in the identification of lines due to bromine itself. The vacuum spectrograph used was of the type designed by Prof. Siegbahn,

in which the grating was mounted at grazing incidence. The dispersion of the instrument is about 3·4 Å./mm. at λ 1400 and about 2·4 Å. mm. at λ 400. This large dispersion is necessary particularly in the case of bromine, which is extremely rich in lines in the vacuum grating region. The plates have been measured with the lines of O, N, C as standards. This comparison spectrum was superposed by changing the lower electrode to one of C or of Al tipped with $LiNO_3$. Details of the method of exciting the spectrum and the method of evaluating the wave-lengths were fully dealt with in a former communication by one of the authors[2]. A portion of the spectrum between λ 650 and λ 750 is reproduced below.

Vacuum spark spectrum of bromine.



(*a*) *Without inductance;* (*b*) *with inductance.*

§ 3. THE SPECTRUM OF Br v

The spectrum of Br v is similar in structure to those of Ga I, Ge II, As III and Se IV and consists essentially of a simple doublet system. The important electron configurations and the resulting term structure are shown in table 1.

Table 1. Term scheme of Br v

3_1	3_2	3_3	4_1	4_2	4_3	4_4	5_1	5_2	5_3	Term prefix	Terms
2	6	10	2	1						$4p$	^2P
2	6	10	2				1			$5s$	^2S
2	6	10	2		1					$4d$	^2D
2	6	10	2			1				$4f$	^2F
2	6	10	1	2						$4p^2$	$^4P\ ^2P\ ^2D\ ^2S$

By the application of the regular and irregular doublet laws to Ga-like atoms it has been possible to establish the important doublet combinations shown in table 2.

Table 2. Classifications in Br v

Combination	λ, with intensity in brackets	ν	δν
$4p\,^2P_2-4p^2\,^2D_2$	855·79 (1)	116851	
$^2P_2-\ \ ^2D_3$	850·79 (8)	117538	*687*
$^2P_1-\ \ ^2D_2$	813·40 (6)	122941	*6090*
$4p\,^2P_2-4p^2\,^2P_1$	657·60 (6)	152068	
$^2P_2-\ \ ^2P_2$	645·49 (10)	154921	*2853*
$^2P_1-\ \ ^2P_1$	632·30 (9)	158153	*6085*
$^2P_1-\ \ ^2P_2$	621·11 (10)	161002	*6081*
$4p\,^2P_2-4d\ \ ^2D_2$	549·81 (6)	181880	
$^2P_2-\ \ ^2D_3$	547·94 (10)	182502	*622*
$^2P_1-\ \ ^2D_2$	532·00 (10)	187970	*6090*
$4p\,^2P_1-5s\ \ ^2S_1$	468·36 (5)	213510	
$^2P_2-\ \ ^2S_1$	482·11 (10)	207422	*6088*

To avoid the use of fractions, an integral inner quantum number greater by half a unit than the true value has been assigned to each term of even multiplicity in the present paper. The above assignments are supported firstly by the behaviour of the lines. It has been definitely ascertained by a careful study of the various plates that they belong to Br v. Secondly the progressions indicated below in table 3 confirm the classifications made.

Table 3. Irregular doublet law applied to corresponding lines

Spectrum	$4p\,^2P_2-4d\,^2D_3$ ν	δν	$4p\,^2P_2-4p^2\,^2D_3$ ν	δν
Ga I	33961		—	
Ge II	79246	*42585*	63412	*19281*
As III	114801	*35555*	82693	*17640*
Se IV	149231	*34430*	100333	*17205*
Br V	182502	*33271*	117538	

Regular doublet law*

Spectrum	$4p\,^2P_1-4p\,^2P_2$	s	δs	$\delta^2 s$
Ga I	826	19·40		
Ge II	1768	17·96	*1·44*	*0·55*
As III	2940	17·07	*0·89*	*0·22*
Se IV	4376	16·40	*0·67*	*0·15*
Br V	6090	15·88	*0·52*	

* Cf. also K. R. Rao and J. S. Badami, *Proc. R.S.* A 131, 166 (1931).

Term values in the spectrum of Br v are given in table 4. The deepest term $4p\,^2P_1$ is given a value zero and the other values are based on this.

Table 4. Term values in Br v

Term	Term value	δν
$4p\ ^2P_1$	0	
$^2P_2$	6090	*6090*
$4p^2\,^2P_1$	158158	
$^2P_2$	161011	*2853*
$4d\ ^2D_2$	187970	
$^2D_3$	188592	*622*
$5s\ ^2S_1$	213510	

§ 4. THE SPECTRUM OF Br VI

The spectrum of Br VI, consisting of triplets and singlets, is isoelectronic with those of Zn I, Ga II, Ge III, As IV and Se V. In all these spectra the chief triplet and singlet terms have been identified. With the aid of the experimental data described in the previous section it has been possible to establish the combinations between the low-lying triplet and singlet terms of Br VI. The combinations identified in the present work are given in table 5.

Table 5. Combinations in Br VI

		$4p\,^3P_0$ 104316	$^3P_1$ 106431	$^3P_2$ 111336	$^1P_1$ 151274
			2115	*4905*	
$4s\,^1S_0$	0		106431 (8)?		151274 (10)
			939·57		*661·95*
$4p^2\,^3P_0$	244136		137711 (6)		
	3270		*726·16*		
$^3P_1$	247406	143100 (6)	140981 (4)	136070 (6)	
	6486	*698·81*	*709·31*	*734·91*	
$^3P_2$	253892		147463 (5)	142560 (8)	
			678·14	*701·46*	
$4d\,^3D_1$	304648	200321 (4)	198210 (6)	193315 (1)	
	319	*499·20*	*504·52*	*517·29*	
$^3D_2$	304967		198532 (8)	193631 (8)	
	484		*503·70*	*516·45*	
$^3D_3$	305451			194115 (8)	
				515·37	

There can be no doubt as to the identification of the two triplet groups. For the choice of the singlet $4s\,^1S_0$–$4p\,^1P_1$, however, an alternative line is possible, namely ν151154 (10), but the behaviour of the two lines under different conditions of excitation favours the selection made in the above scheme, for the line $^1S_0$–$^1P_1$ is a resonance line and must appear under the weakest stimulus.

Table 6, which shows the application of the regular and irregular doublet laws to Zn-like spectra, supports the correctness of the above identifications. This table is an extension of similar sequences by Sawyer and Humphreys[3].

Table 6. Regular doublet law

Term	Spectrum	$\delta\nu$	s	δs	$\delta^2 s$
$4p\,^3P$	Zn I	579·2	19·38		
	Ga II	1382	17·80	*1·58*	0·63
	Ge III	2403	16·85	*0·95*	*0·25*
	As IV	3677	16·15	*0·70*	*0·16*
	Se V	5210	15·61	*0·54*	*0·12*
	Br VI	7020	15·19	*0·42*	
$4d\,^3D$	Zn I	7·95	25·22		
	Ga II	65	22·91	*2·31*	0·96
	Ge III	179	21·58	*1·33*	*0·55*
	As IV	336	20·80	*0·78*	*0·18*
	Se V	550	20·20	*0·60*	*0·23*
	Br VI	803	19·83	*0·37*	

Irregular doublet law

Spectrum	$4p\,^3P_2-4p^2\,^3P_1$	$\delta\nu$	$\delta^2\nu$
Zn I	—		
Ga II	66472	*18034*	
Ge III	84506	*17475*	*559*
As IV	101981	*17139*	*336*
Se V	119120	*16950*	*189*
Br VI	136070		

In table 7 are given the term values in Br VI, based on the deepest term $4s\,^1S_0$ as zero.

Table 7. Term values in Br VI

Term	Term value	$\delta\nu$
$4s\,^1S_0$	0	
$4p\,^3P_0$	104316	*2115*
$4p\,^3P_1$	106431	*4905*
$4p\,^3P_2$	111336	
$4p\,^1P_1$	151274	
$4p^2\,^3P_0$	244136	*3270*
$^3P_1$	247406	*6486*
$^3P_2$	253892	
$4d\,^3D_1$	304648	*319*
$^3D_2$	304967	*484*
$^3D_3$	305451	

§ 5. THE SPECTRUM OF Br VII

The spectrum of Br VII like that of As V consists of simple doublets arising from the transition of the outermost s electron to the various excited states. Two pairs resulting from a combination between the low-lying doublet terms, $4s\,^2S$, $4p\,^2P$ and $4d\,^2D$ have been discovered. It was first definitely ascertained that the lines forming these combinations belonged to Br VII, by a close scrutiny of the plates. The pairs identified are given in table 8. The pair $4p\,^2P-5s\,^2S$ is expected to lie beyond the region investigated.

Table 8. Doublets of Br VII

Classification	λ (and intensity)	ν	$\delta\nu$	Term values
$4s\,^2S_1-4p\,^2P_1$	779·58 (8)	128274	*7580*	$4s\,^2S_1 = 0$
$4s\,^2S_1-4p\,^2P_2$	736·09 (10)	135854		$4p\,^2P_1 = 128274$
$4p\,^2P_2-4d\,^2D_2$	522·60 (0)	191351	*861*	$^2P_2 = 135854$
$4p\,^2P_2-4d\,^2D_3$	520·26 (4)	192212	*7579*	$4d\,^2D_2 = 327205$
$4p\,^2P_1-4d\,^2D_2$	502·69 (4)	198930		$^2D_3 = 328066$

As in the previous spectra, recourse has been had to the method of application of the X-ray laws to the sequence of Cu-like spectra to which the spectrum of Br VII

belongs. These regular progressions which are exhibited in the following table constitute excellent evidence of the correctness of the identification.

Table 9. Regular doublet law

Spectrum	$4p\,^2P_1-4p\,^2P_2$	s	δs	$\delta^2 s$
Cu I	248	20·19		
			1·96	
Zn II	873	18·23		*0·79*
			1·17	
Ga III	1718	17·06		*0·38*
			0·79	
Ge IV	2790	16·27		*0·19*
			0·60	
As V	4110	15·67		*0·12*
			0·48	
Se VI	5700	15·19		*0·10*
			0·38	
Br VII	7580	14·81		

Irregular doublet law

Spectrum	$4s\,^2S_1-4p\,^2P_1$	$\delta\nu$	$\delta^2\nu$
Cu I	30535		
		17949	
Zn II	48483		*1264*
		16684	
Ga III	65167		*544*
		16140	
Ge IV	81307		*312*
		15828	
As V	97135		*201*
		15627	
Se VI	112762		*115*
		15512	
Br VII	128274		

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ON THE TILTED ELECTROMETER

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ABSTRACT. A detailed description is given of the construction and performance of a new, evacuated, critically damped, quick-reading, quartz-fibre electrometer. The sensitivity obtainable is limited only by the Brownian motion of the fibre. Photographic records are reproduced showing the response of the electrometer to suddenly applied potential-changes, and to the ionization currents of individual α particles. The minimum potential-change that can be measured (with the usual convention that the corresponding deflection of the system be not less than four times the root of mean square of the deflections of the Brownian motion) is of the order of 0·0001 V. when the period (undamped) is 5 seconds, and 0·0005 V. when the period is 1 second. The range of approximately constant sensitivity is adequate for most purposes.

§ 1. INTRODUCTION

An improved model of the Wilson tilted electrometer[1],[2] has been described in a previous paper[3]. This instrument had an outer airtight protecting case, and the design was arranged so that the tilting of the electrometer, the readjustment of the telescope, and the movements of the hanging leaf or fibre all took place about the same axis. It was stated that the instrument was reliable and very convenient to operate, that the range of approximately constant sensitivity and the stability of the instrument were for a given sensitivity much greater with a silvered quartz fibre hinged to the electrode by a piece of gold leaf than with a complete gold leaf, and that a somewhat higher voltage-sensitivity could be obtained with a fibre than with a gold leaf; but that the instrument responded inconveniently slowly at high sensitivity with either suspension. It was shown on filling the electrometer with hydrogen instead of air that this sluggishness was caused by the damping due to the viscous resistance of the gas.

An investigation has now been completed of the behaviour of this type of electrometer with the air damping of the fibre reduced and adjusted by a controlled evacuation of the air in the electrometer[3]*.

It has been found, with a gold-sputtered quartz fibre of diameter about 10^{-4} cm. in the evacuated electrometer, that very high voltage sensitivities may be obtained, and that, when the motion of the fibre is critically damped, the time taken by the electrometer in responding to a potential change is remarkably short. The magnification that can with advantage be used is usually limited only by the Brownian motion of the fibre. The gold-leaf hinge which was inserted in the electrometer

* See the discussion, p. 185.

described in the previous paper is unnecessary with very fine quartz fibres, and is in fact disadvantageous on account of its comparative rigidity.

A new design for an electrometer, having all the advantages of the original one and in addition the advantage of simple construction, will first be described, and an account will then be given of the behaviour of the evacuated electrometer and of some essential details of the method of preparing and inserting suitable fibres.

It is evident that for rapid readings in the direct measurement of small electric charges the electrometer is superior to any of the well-known instruments. It is superior also in its general simplicity, and in the ease and rapidity with which its sensitivity may be changed.

§ 2. DESIGN

The electrometer, figure 1, was made of brass tube and plate of standard sizes, no castings being used. The joint E in the outer case, the quartz insulator F, and the glass insulator G, were made airtight with compressed rubber. The windows H were closed with good microscope cover glass sealed in position with a vacuum wax. The windows of the inner case could be left without glass but the fibre was

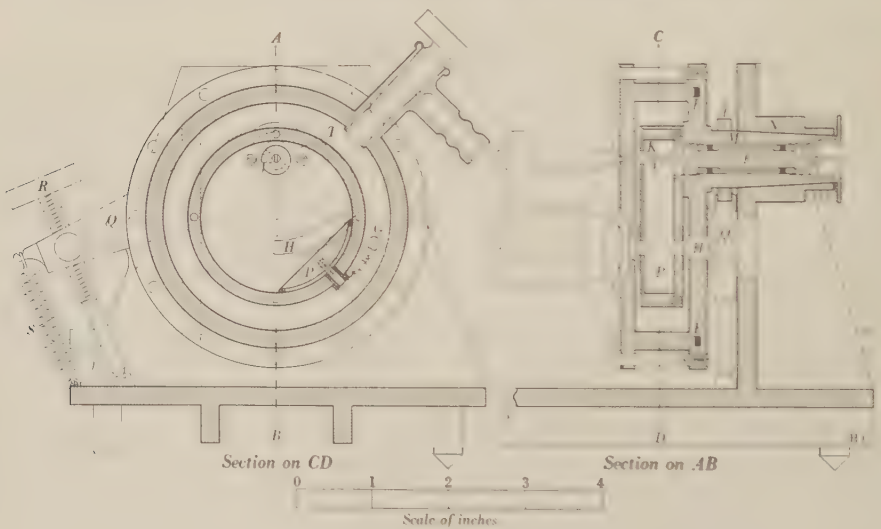

Figure 1.

then more liable to damage from any sudden change of the air pressure in the instrument. The employment of both an inner and an outer case was found to be absolutely essential for adequate protection of the fibre, even when the electrometer was evacuated. The whole instrument was supported by the axle M in the bearing N, and the tilt was adjusted with a long rigid lever Q, which was clamped to the axle. The end of this lever was controlled by a milled head screw R and a strong spring S. A preliminary rough adjustment of the tilt was obtainable by slackening the clamp T.

The positions of the plate P, and the point of suspension V of the fibre in the

electrometer case, were to a certain extent arbitrary, and for the purpose of reproduction were chosen so that the plate formed one side of a square inscribed in the case, and the axis V was on the diagonal of the square $\frac{1}{8}$ of its length from one end. Any improvements in the performance of the electrometer made by the use of a different disposition of the plate and of the fibre and by the introduction of a curved plate were found to be of quite secondary importance. The important factor was the type of fibre used (see below).

The best length for a given fibre in the electrometer, as was found by Kaye for gold leaves[2], was the maximum length that could be employed without danger of the fibre touching the plate—in this model about 3 cm. Shorter fibres require higher plate voltages.

It will be seen that this electrometer is suitable for mounting directly through the wall of an ionization chamber by incorporating the bearing N, suitably modified, in the chamber wall.

The work about to be described was not carried out with this improved electrometer but with the original model of the instrument[3], the only important difference being that the inner case of the older model was $\frac{1}{8}$ in. wider.

§ 3. GENERAL PERFORMANCE

The evacuation of the electrometer was found to produce no great improvement in its performance with a gold-leaf suspension. The time taken by the leaf to move to a new position of rest was, it is true, greatly reduced at the higher sensitivities, but the sensitivity of the electrometer was limited not so much by the damping of the leaf as by the extreme sensitiveness of the leaf to change in the tilt which is a characteristic of the electrometer with all suspensions when the range of approximately constant sensitivity becomes unduly short. Thus a typical leaf, using a microscope magnification such that 53 eyepiece scale-divisions corresponded to 1 mm., had a sensitivity of 500 divisions per volt over a range of only 4 or 5 divisions, and was then much too easily affected by any slight change in the level of the electrometer or in the shape of the leaf to have a constant zero-position. This sensitivity is about the maximum to be obtained with a gold leaf whether the electrometer is evacuated or not.

Further, very few of the gold leaves used were free from irregularities in their calibration curves. These irregularities, which must be well known to those who have used gold-leaf electroscopes, were either irregular changes in the slope of the curve or actual breaks where the leaf jumped suddenly to a new position without change in its potential. When the leaf was brought back, by means of a continuously variable potentiometer, over a range having one of these breaks, the return jump did not usually occur at the same leaf-potential as the former but somewhat later, so that the calibration curve had a kind of hysteresis loop. In regions removed from a discontinuity no hysteresis effects were observed with gold leaf.

A general description will now be given of the behaviour of the electrometer with a particular quartz fibre, which will be called "fibre no. 1." This fibre was

about 10^{-4} cm. in diameter and comparatively heavily (see below) sputtered with gold. It is to be noted that it was a fibre of medium quality, exemplifying both the advantages and some of the possible faults of these quartz-fibre suspensions.

When the sensitivity was adjusted to about 2000 divisions per volt, with the above-mentioned microscopic magnification, the instrument was found to be of little practical use when the enclosed air was at atmospheric pressure, because the fibre took many minutes to reach its new position of rest when a change of potential occurred. With a pressure of air in the electrometer, however, of 0·03 mm. of mercury, the motion of the fibre was critically damped, and it then took little more than 1 second to reach its new position. With a still lower air pressure in the electrometer the fibre oscillated before coming to rest. The most suitable air pressure to have in the electrometer at any sensitivity for most purposes is, of course, that at which the fibre is critically damped.

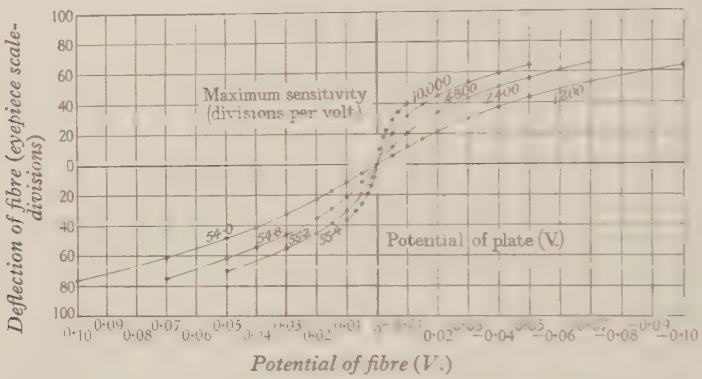

Figure 2. Fibre no. 1. Calibration curves at different plate-potentials.

The electrometer did not become unduly affected by slight changes in level till the sensitivity of the fibre had been raised to 10,000 divisions per volt, a sensitivity which is 20 times as great as that obtained with gold leaf. For the sake of completeness the calibration curves of fibre no. 1 are here given in figure 2. The calibration of the electrometer will be dealt with in detail in another section.

Another factor, which may have been either an elastic hysteresis or an electrical after-effect, influenced the performance of this fibre. The statement made above, that at 2000 divisions per volt the fibre took little more than 1 second to reach a new position of rest must be modified. It was found that a typical complete deflection of the fibre consisted of two parts, (1) a sudden initial deflection taking about 1 second, and (2) a creep over a small extra distance continuing for 2 or 3 minutes. This creep was least when the fibre was brought back to a potential from which it had been only momentarily changed, and was greater for a large deflection than for a small one. It was most troublesome at high sensitivity. It has been found (see below) that this defect in the performance of a sputtered quartz fibre in this electrometer can be entirely eliminated by sufficiently reducing the thickness

of the gold coating. It is of interest to notice that an after-effect very like this has been investigated by Israël[4] in 12 models of the Lindemann electrometer[5], manufactured by Spindler and Hoyer, Göttingen. He found that the needle after its first sudden excursion approached its final position exponentially and was usually very near to it after 5 minutes. He does not in that paper suggest a remedy for the fault.

A Cambridge falling-plate camera was now used to study the behaviour of this fibre. An image of the fibre was formed by a cinema projecting-lens with condensed light from a pointolite lamp. The beam of light on its way to the camera was reflected successively by two plane mirrors so that the shadow of the fibre on the front of the camera was vertical. Lines engraved at intervals of 1 mm. on the cylindrical lens in front of the camera slit produced a scale on the plates for displacements of the fibre, and a pendulum, near the lens, which cut off the light once a second, gave a time scale. The magnification was adjusted to be 53:1.

The manner in which the fibre oscillated about a new position of equilibrium, when the air pressure in the electrometer was much less than that required for critical damping of the fibre, is shown in figure 3, and the response of the critically damped fibre to suddenly applied potential-changes in figure 4. Thus, in figure 4, the trace at 500 mm./V. shows for the first 5 seconds the position of the earthed fibre. Then the potential of the fibre was changed to $-0{\cdot}01$ V. and in less than 1 second the image moved about 5 mm. to a new position of rest. After 5 seconds the fibre was again earthed. Then $-0{\cdot}02$ V. was applied for 5 seconds, and so on. The trace at 5000 mm./V. shows the adverse effect of the elastic hysteresis of this fibre.

The period of the fibre at different sensitivities was roughly measured from a series of photographic records of the type reproduced in figure 3. The air pressure in the electrometer was the same throughout the series, and was not low enough to allow many oscillations of the fibre at the higher sensitivities, but no correction was made for the effect of the damping on the period. The results are plotted in the upper curve in figure 5, which shows the measured period against the square root of the sensitivity. Substitution of the undamped period would make the curve more nearly linear.

The air pressures required for critical damping at different sensitivities of the electrometer were adjusted by running the Cenco Hyvac pump, which was used to evacuate the electrometer, against an air leak through a fine glass tube, and were measured by a Macleod gauge. A mercury-vapour trap was incorporated between the gauge and the system. The pressures ranged from 0·07 mm. of mercury at 500 mm./V. to 0·019 mm. at 5000 mm./V. When the logarithms of the pressures were plotted against the logarithms of the sensitivities all the points lay almost exactly on the same straight line. It will be observed that, on account of this change in the air pressure required for critical damping of the fibre at different sensitivities, the trace at the higher sensitivity in figure 5, where the range of constant sensitivity was not great (see figure 2), shows that the fibre was not critically damped for large deflections from zero.

There was a marked radiometer force on the fibre in the evacuated electrometer, particularly in the case of the strong illumination used for these photographs. If the pressure of the air and the intensity of the light were kept constant, however, the radiometer force had no adverse effect on the performance of the fibre. The trace reproduced in figure 6 shows the displacements of the zero position of the

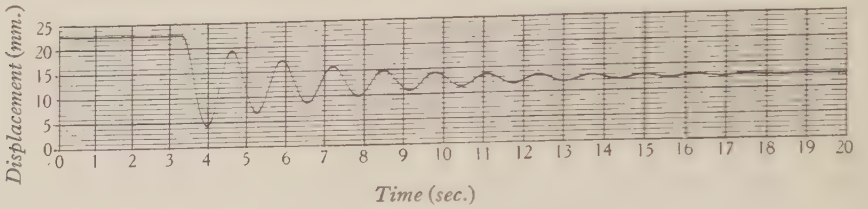

Figure 3. Period at 500 mm./V.

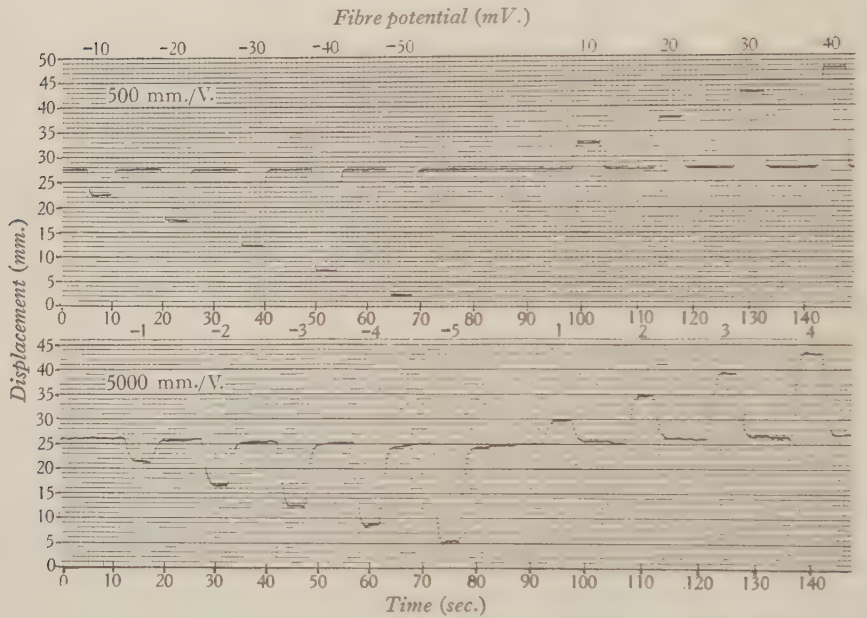

Figure 4. Response of the electrometer, fibre no. 1, to sudden potential-changes. The trace at 5000 mm./V. shows the adverse effect of sputtering a fibre too heavily.

earthed fibre, at a sensitivity of 2000 mm. V., brought about by putting pieces of plate glass suddenly into the beam from the pointolite lamp. The response of the instrument to a change in the intensity of the light is remarkably sudden.

In order to investigate the hysteresis found with fibre no. 1, and also in order to find whether the lightness of a quartz fibre was the factor which accounted for its superiority to gold leaf in the electrometer, experiments were made with a number of differently sputtered fibres. The result of these experiments can be deduced from the behaviour of the three fibres by which the experiments were

finally corroborated. These fibres will for reference be numbered 2, 3 and 4. They were cut from the same piece of quartz fibre of diameter between 0·001 and 0·002 mm., and were prepared as follows. Fibre no. 2 was sputtered for 1 hour. Fibre no. 3 was sputtered for 2 minutes over its whole length, and then for 1 hour over 1 cm. of its length, the rest being shielded. Fibre no. 4 was sputtered for 2 minutes only. The rate of sputtering was such that a coating of gold, which seemed to the eye to be as opaque as ordinary gold leaf, i.e. somewhat less than

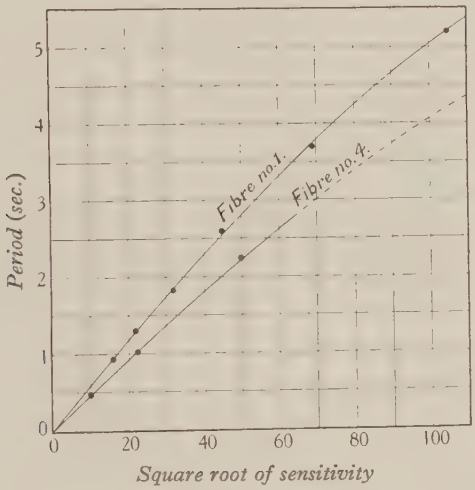

Figure 5.

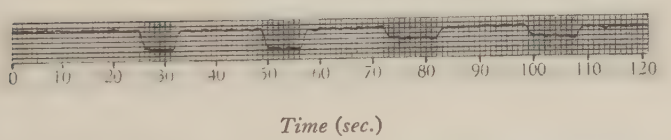

Figure 6. The radiometer effect: changes in the position of the earthed fibre produced by reducing the intensity of illumination with pieces of plate glass. Sensitivity 2000 mm./V.

10^{-5} cm. in thickness, was deposited in 1 hour on a flat glass plate placed in the sputtering apparatus.

The heavily sputtered fibre no. 2 was found to be quite useless. Its calibration curves, as far as they could be determined, corresponded to those of a gold leaf, and it had a comparatively quick initial response to a potential change, but the hysteresis effects were so great that it sometimes had to be kept near one position for more than a day before it stopped creeping.

Fibre no. 3 was inserted so that the heavily sputtered length was at the lower end of the suspension. The upper two centimetres, particularly the part near the electrode where most of the bending must occur, had thus a very thin coating of gold. No hysteresis was observed with this fibre, and in addition it responded

quite sharply to a sudden potential-change of the supporting electrode, which showed that the electrical resistance of the part of the fibre near the electrode was not too great, despite the thinness of the gold coating. On the other hand, on account of its weight, this fibre was inferior to a gold leaf in that the maximum useful sensitivity was less than 400 divisions per volt!

Fibre no. 4 was used for 4 months in the electrometer, and proved to be very satisfactory. The maximum sensitivity was found to be of the order of 12,000 eye-piece scale-divisions per volt (figure 9), and there were no hysteresis effects. The period was less than that of fibre no. 1, at any given sensitivity of the fibres, in the ratio of approximately 4:5 (figure 5), and was very much less than that of fibre no. 3 or that of a gold leaf.

As these three fibres were adjacent lengths of the same piece of fibre, and so would have performed similarly in the electrometer had they been similarly sputtered, the following conclusions can be drawn: (i) The after-effect with gold-sputtered fibres is caused by the gold coating and disappears if a thin enough film of gold is used; (ii) in this electrometer high sensitivity is obtainable when the weight of the moving part is small; and (iii) the period of the electrometer at a given voltage-sensitivity of the fibre is reduced by employing a lighter fibre. The behaviour of fibre no. 2 suggests that even the creep of long duration, often experienced at high sensitivity with the sensitive electrometers which employ a quartz fibre in torsion, may sometimes be caused by the metal film on the fibre.

§ 4. ADJUSTMENT AND CALIBRATION

V The deflection of the fibre by alteration of the tilt at a fixed plate-potential V will now be considered, the fibre and the case of the electrometer being at zero θ, ϕ potential. If θ is the angle between the fibre and the vertical, and ϕ is the angle between the plate and the horizontal, as marked in figure 7 (*a*), then when ϕ is

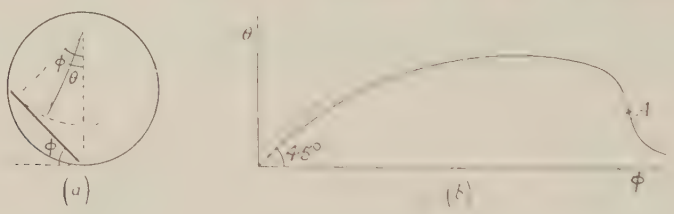

Figure 7.

small θ is approximately equal to ϕ, but as ϕ is increased the fibre tends to fall away from the plate and θ becomes less than ϕ. If the voltage V is not too great, θ reaches a maximum as ϕ is increased, after which it decreases more or less rapidly to small values, figure 7 (*b*). The fibre is found to be most sensitive to a change in its own potential at a point A on this curve where it is also most sensitive to a change in tilt. In using the electrometer, therefore, the fibre should be brought, by altering the

tilt, to the point A on one of these curves where the sensitivity of the fibre is constant over a certain range and is a function of the plate-voltage only. The deflection of the fibre at the point A will be called θ_A.

θ_A

Two systems of these θ, ϕ curves are given in figure 8, one with plate-voltages 62, 64, 66 and 68 for fibre no. 3, and the other with plate-voltages 34, 36, 38 and 40 for fibre no. 4. The values of the tilt and of the plate-voltage in one of these systems of curves must not be too closely compared with the values in the other because these quantities are greatly dependent on the length of a fibre. The end of fibre no. 4 had an inherent curl, which was straightened out when the distance by which it cleared the plate was adjusted, and so it was effectively shorter than fibre no. 3. With a straight fibre, sputtered like fibre no. 4 and of rather smaller diameter, instability has been obtained with a plate potential of only 27 volts. Still finer quartz fibres could probably be prepared and inserted.

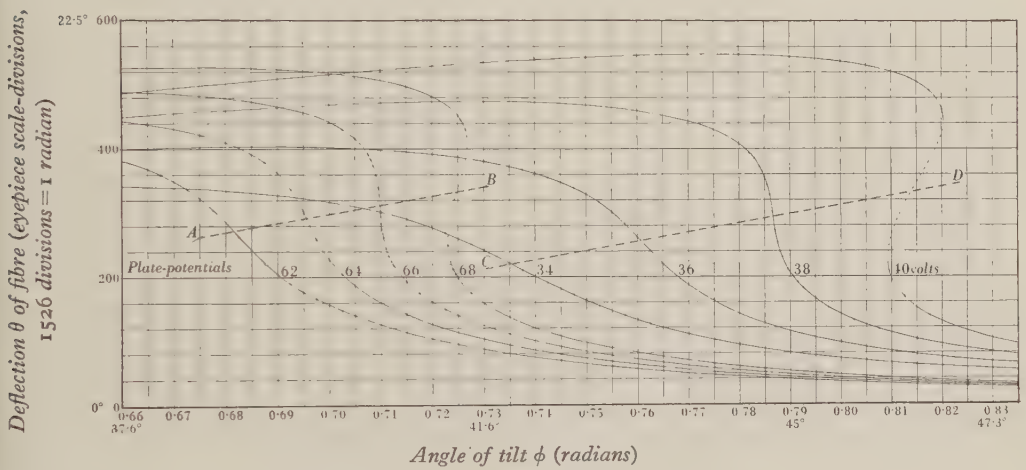

Figure 8.

The broken lines AB and CD, drawn through the points of greatest slope on these curves, are the loci of the deflections θ_A of the fibre, and of the positions of the microscope for maximum sensitivity in the middle of the eyepiece scale. If, however, the change in sensitivity is not great, so that θ_A does not change much, then the microscope does not require to be readjusted, and the operation of changing the sensitivity of the electrometer entails merely (i) alteration of the plate-potential to the value appropriate to the new sensitivity, and (ii) a corresponding change of tilt to bring the image of the zero potential fibre back to the middle of the microscope scale. The calibration curves in figures 2 and 9 have been taken with the microscope in a fixed position, and so show the magnitude of the error in the setting of the microscope to be expected when the sensitivity is changed in this way.

The curve at the highest plate-potential in each of the systems in figure 8 has a range of ϕ for which three values of θ are possible. At one of these deflections the fibre is in unstable equilibrium. The transition curve will have a critical point at which, theoretically, the potential sensitivity of the fibre will be infinitely great.

In figure 9 the fibre-potential calibration curves of fibre no. 4 have been treated in some detail. Three subsidiary graphs are given, showing (*a*) the change in the angle of tilt with sensitivity, (*b*) the change in the plate-potential with sensitivity, and (*c*) the length of the linear scale at different sensitivities. Curves (*a*) and (*b*) refer to the electrometer with the zero-potential fibre at the centre of the microscope scale, where all the curves intersect, and the changes in the tilt and in the plate-potential have been measured from the values at the critical point where the potential-sensitivity of the fibre would be infinite. To obtain curve (*c*) a "linear scale" was defined in the following way. A tangent to each calibration curve was drawn through its point of inflexion, and the length of the linear scale was taken to be such that at its ends the calibration curve differed from this tangent by 10 per cent of the displacement of the fibre from the point of inflection. The length of the linear scale was greater with fibre no. 4 at a sensitivity of 12,000 divisions per volt than with fibre no. 3 at 400 divisions per volt.

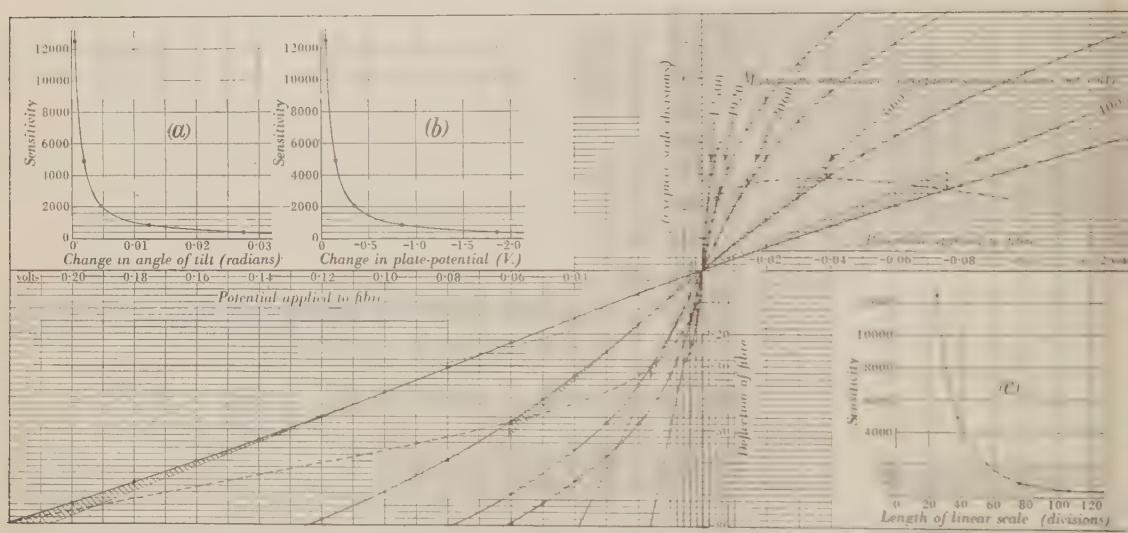

Figure 9.

The above calibration was carried out with a rather low optical magnification of the deflections of the fibre, namely 53 eyepiece scale-divisions to 1 mm., so that a large range of the sensitivity curves could be obtained without moving the microscope. The degree to which the sensitivity of the electrometer could be increased by the use of increased optical magnification of the fibre will be discussed in the next section.

The relation between the fibre-potential and the plate-potential sensitivity of the electrometer may be given. If the same small deflection of the fibre $\delta\theta$ is produced by a change of fibre-potential δV_F with the plate at constant potential V_P, and by a change of plate-potential δV_P with the fibre at constant potential V_F, it is found that $2\delta V_F = -\delta V_P$.

§ 5. LIMITING SENSITIVITY

It will be observed in figure 4, particularly at the higher sensitivity, that the fibre (no. 1) continually executed an irregular vibration about its mean zero position. It is evidently this motion of the fibre that imposes a limit upon the increase in sensitivity obtainable by increase of the optical magnification of the deflections of the fibre. The motion was not in the least reduced by placing the electrometer on a very stable concrete basement floor. It also remained unchanged in magnitude day and night, so that it could not have been caused by traffic vibrations from the streets. An irregular motion of the same kind, in the suspended system of the Hoffmann vacuum duanten-electrometer[6], has been shown by Eggers[10] to be the Brownian motion of the system, by the method of making from it a statistical calculation of Boltzmann's constant.

The equation of a harmonically bound system in Brownian motion is

$$R\bar{\phi}^2 = kT \text{ [8], [9]} \qquad \text{.....(1),}$$

where k is Boltzmann's constant, T the absolute temperature, R the restoring couple of the system per unit angular displacement, and $\bar{\phi}^2$ the mean square angular deflection of the system from its mean position. In the absence of a mathematical theory of the electrometer analogous to that[10] which Eggers used to enable him to evaluate R for substitution in this formula, it is not possible to repeat his work with the electrometer. The restoring couple R can, however, be eliminated, if the period P and the moment of inertia I of the fibre are introduced by the equation

k, T, R

$\bar{\phi}$

P, I

$$P = 2\pi\sqrt{\frac{I}{R}} \qquad \text{.....(2).}$$

The resulting equation is

$$\sqrt{\bar{\phi}^2} = \frac{\sqrt{(kT)}}{2\pi}\frac{P}{\sqrt{I}} \qquad \text{.....(3).}$$

By means of equation (3) it was very readily found that the above-mentioned motion of the fibre was the Brownian motion.

A rough measure of $\sqrt{\bar{\phi}^2}$ was obtained at a sensitivity of the electrometer of 2500 eyepiece scale-divisions per volt, with fibre no. 4, the microscope being adjusted so that 53 divisions corresponded to 1 mm. The readings of the position of the fibre, from which $\sqrt{\bar{\phi}^2}$ was calculated, were taken at intervals of 5 seconds, the magnification being increased by receiving an image of the fibre and the eyepiece scale on a graduated screen placed some distance away from the ocular of the microscope.

The value of $\sqrt{\bar{\phi}^2}$ was $1{\cdot}0 \times 10^{-4}$ radian, the period 2·2 seconds, and the absolute temperature 293° C. Using $1{\cdot}4 \times 10^{-16}$ erg/°C. per degree of freedom as the value of Boltzmann's constant, and substituting in equation (3), it is found that I is $5{\cdot}0 \times 10^{-7}$ gm. cm^2. Now this value of the moment of inertia of the fibre is the value obtained by assuming that the diameter is $1{\cdot}5 \times 10^{-4}$ cm., and that the mean density of the quartz and gold is 3 gm./cm^3. A direct measurement of the diameter of the fibre was not made, but the fibre was taken from a batch of diameter "about

10^{-4} cm." supplied by the Cambridge Instrument Company, so that it probably was not in fact of much greater diameter than $1{\cdot}5 \cdot 10^{-4}$ cm. It is therefore evident that the greater part at least of the energy of this irregular motion of the fibre must be the equipartition energy of the Brownian motion. If the moment of inertia of the sputtered fibre were determined with some accuracy a measurement of Boltzmann's constant could readily be made by this method.

The minimum displacement which can be measured, under suitable magnification, in the case of a system executing the Brownian motion, is usually taken to be four times the root mean square of the Brownian displacements. The minimum voltage-change which can be read with the electrometer will therefore be, according
S to this convention, $4\sqrt{\bar{\phi}^2}/S$, where S is the voltage-sensitivity of the fibre in angular deflection per volt. Now it has been shown experimentally, figure 5, that the period of the fibre is, for a given fibre, approximately proportional to the square root of the sensitivity, i.e.

$$P \doteqdot A\sqrt{S} \qquad \text{.....(4),}$$

A where A is a constant. Therefore combining equations (3) and (4) we find that

$$\frac{4\sqrt{\bar{\phi}^2}}{S} \doteqdot \frac{4A^2\sqrt{(kT)}}{2\pi\sqrt{I}} \cdot \frac{1}{P} = \frac{B}{P} \qquad \text{.....(5),}$$

B where B is constant for a given fibre at a given temperature. Equation (5) shows that the minimum voltage which can be measured by the electrometer with a given suspension and sufficient optical magnification is approximately inversely proportional to the period.

It will be convenient, for purposes of comparison in the following table, to
L define a "limiting sensitivity" L of the electrometer such that

$$L = S/4\sqrt{\bar{\phi}^2} \qquad \text{.....(6).}$$

Table

Undamped period P (sec.)	Sensitivity S (mm./V.) Magnification × 53	Limiting sensitivity L $\left(\frac{S}{4\sqrt{\bar{\phi}^2}}/\text{V.}\right)$	Magnification for $4\sqrt{\bar{\phi}^2}$ to give 1 mm.	10 % linear scale (mm.)
0·45	100	890	× 450	—
1·0	500	1,970	× 200	360
2·2	2,500	4,340	× 90	90
5·0	12,500	9,860	× 42	20

Thus L is the reciprocal of the minimum voltage which can be measured. The limiting sensitivity, with any given fibre, will be proportional to the period.

Some values of the limiting sensitivity of the electrometer with fibre no. 4 over a range of period from 0·45 to 5 seconds, are given in the table. The values were calculated according to equation (5) from the rough measurement of $\sqrt{\bar{\phi}^2}$ at the period 2·2 seconds. The calculated values could not be properly verified experimentally at the two extremes of the range, on account of the difficulty of obtaining sufficient magnification at one end, and on account of the unsteadiness of the fibre

from causes other than the Brownian motion at the other end. It must be emphasized that near its maximum sensitivity the electrometer is of use only as a detector, unless very special precautions are taken to keep the tilt and the plate-voltage constant.

The table gives also the optical magnification that would be necessary to make $4\sqrt{\bar{\phi}^2}$ correspond to a deflection of the image of the fibre amounting to 1 mm., and the length of the 10 per cent linear scale in mm. with this magnification. There is no doubt that it would be quite possible, with a lighter fibre in the electrometer, to reduce the restoring couple sufficiently for the Brownian motion to be observed with unit magnification, or in other words with the naked eye.

§ 6. MEASUREMENT OF CHARGE

No direct measurement has yet been made of the charge-sensitivity of the electrometer, but because of the very small capacity of the fibre itself, and the high potential-sensitivities which have been obtained, it is clear that the maximum charge-sensitivity must be high. It is probable that a charge of the order of magnitude of 1000 electrons could be detected without difficulty. If, in order to obtain a high charge-sensitivity, the electrometer were to be used in connection with an insulated system of very small capacity, it would be advantageous to modify the design given in figure 1 so as to reduce the dimensions of the electrode as much as possible. The bearing N, figure 1, could be arranged if necessary in front of the electrometer, a much shorter electrode being taken through the back.

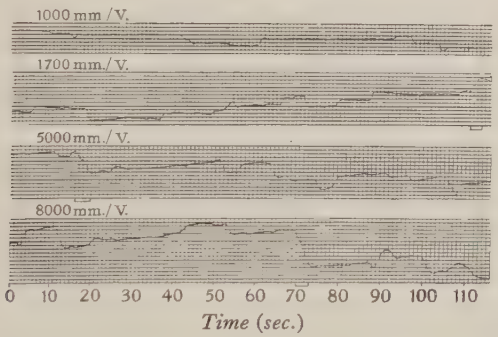

Figure 10. Potential-variations due to the natural leaks in two opposed ionization chambers filled with hydrogen, showing the ionizations due to individual α particles.

In figure 10 are some photographic records of the response of the electrometer (fibre no. 1) to the sudden ionization currents caused by α particles passing through hydrogen. The electrometer was connected to an insulated electrode of comparatively large capacity (total capacity of system 18 cm.) which passed into each of two ionization chambers. The cases of these were at potentials of + 300 and − 300 V. respectively. Traces of radioactive substances, which are always by nature present on the walls of the chambers and in the enclosed gas, gave rise to α particles, the more energetic of which would each produce about 10^5 pairs of ions in their passage through the gas. The currents across the chambers, resulting from these sudden

ionizations, were directed towards or away from the electrode according as the α particle occurred in the chamber with its case at the positive potential or in the other, and the sudden changes in position of the fibre, corresponding to the α particles, were on the average as frequent in the one direction as in the other. The sudden changes of position of the fibre were of the right order of magnitude to have been caused by ionization due to α particles, and it is not possible that they were caused otherwise than by ionization, because (i) after the potential-gradient across the ionization chambers had been removed, no large changes of position of the fibre were observed; and (ii) the sudden changes did not cease to occur when a resistance-capacity smoothing circuit, having a time-constant of about 50 seconds, was interposed between each ionization chamber and the battery used to maintain its potential. The movements of the fibre were found to be more sudden and of greater magnitude with hydrogen in the chambers than with air, carbon dioxide, or sulphur dioxide. These differences are probably accounted for by the different mobilities of the ions in the respective gases, and by greater recombination of ions in the heavier gases. The Hoffmann duanten-electrometer has been used by Ziegert[11] to measure the ionizations of individual α particles. The time taken by his instrument to register an α particle was, however, about 10 times as long as that required by this new electrometer.

When the ionization chambers described in the preceding paragraph were irradiated with X rays or with γ rays in such a way that the opposed ionization currents were on the average equal in magnitude, a very remarkable motion of the fibre of the electrometer was observed, similar in character to the Brownian motion of the fibre, but of much greater magnitude, except with very weak radiation. It was in fact easy to have the excursions of the fibre occupy the whole field of view of the microscope. No exact measurements have been made, but it is quite certain that this effect is due principally to the statistical fluctuations in the ejection of photoelectrons by the radiation. If, for example, it be assumed that each photoelectron ejected by a given radiation in an ionization chamber produces 10^3 pairs of ions in the gas in the chamber, and that the intensity of the radiation traversing the chamber is such that on the average 10^4 photoelectrons per second are ejected in the chamber, then the average current across the chamber is 10^7 electron charges per second and the root-mean-square fluctuation in this current, on account of the statistical fluctuation in the number of photoelectrons ejected by the radiation per second, will be $10^3 \times \sqrt{10^4}$ or 10^5 electrons per second, an amount of charge about equal to the ionization of an α particle and easily detected by the electrometer. This fluctuation will occur in both chambers, and the average currents being balanced out, the electrometer shows only the superposed fluctuations. As would be expected, the movements of the fibre become more vigorous as the intensity of the radiation is increased, and with X rays of different hardness but giving the same ionization currents in the chambers the fluctuations are greater for the harder rays.

All the well-known torsion electrometers, the Hoffmann[6], the Compton[12], the Lindemann[5], and the Perucca[13], employ a needle the moment of inertia of which cannot be reduced below a certain minimum, in view of the need for rigidity.

The great advantage of the principle of the tilted electrometer is that there is no such limit to reduction of the inertia of a hanging fibre, so that it enables a much shorter period at high sensitivity to be reached. A better performance, particularly in the measurement of charge, might be obtained by the use of a vertical fibre with two oppositely charged plates, either parallel to the fibre as in the string electrometer or oppositely inclined as in the Bumstead modification of the Wilson electrometer[14], but simplicity, which is a feature of the Wilson electrometer, would then be lost.

§ 7. MANIPULATIVE DETAILS

The operation of inserting a fibre into the electrometer is difficult if a proper method is not followed, but the method which will now be described enables a new fibre to be mounted on a holder, sputtered, and inserted into the electrometer in less than an hour. There is no doubt that even simpler methods could be devised. The essential point in any method is that the lower end of the fibre be controlled until the inner case of the electrometer has been closed, and then set free after the fibre has been protected from air currents by the inner case. It is not possible to manipulate a freely hanging length of fine quartz fibre as one would a gold leaf.

A convenient form of holder for the fibre is shown in figure 11. The flat-sided spindle *K*, figures 1 and 11, is taken from the electrometer and fitted firmly into the holder as shown. Shellac to fix the fibre is applied at *K* and *L*, figure 11. The fibre is then brought up to the holder and the shellac softened by heating the holder with a small flame. Finally a small paper weight *M*, about 1 mm. square, is attached with seccotine near one end of the fibre. The distance *KM* is about 3·5 cm. Suitable fibres are supplied in 20-cm. lengths by the Cambridge Instrument Company. Four of these holders can be filled at one time by mounting them in line and bringing them into contact with a 20-cm. fibre.

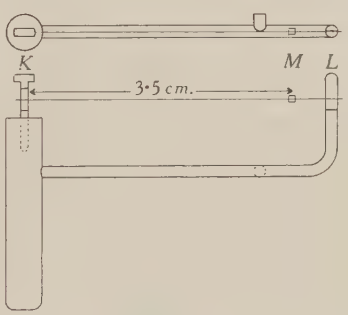

Figure 11.

For sputtering, the holder with its fibre is laid on a glass plate and placed under the gold plate of the sputtering apparatus. Then, after the holder has been fixed in a vice so that the fibre is vertical, the fibre is cut between *L* and *M* with a small pair of scissors, and the spindle *K* with the attached hanging fibre and paper weight is transferred to the electrometer. After the inner case of the electrometer has been screwed up, all that remains to be done is to cut the fibre to the correct length, allowing the paper weight to fall to the bottom of the case where it does no harm. The cutting is best done with a small pair of scissors through the front windows of the case. The plate should be charged to about 60 V. so that it will control the fibre after the latter has been cut. The lower edge of the window serves as a rest for the blades of the scissors and as a guide for the length of the fibre. The cut fibre when pointing directly to the plate should clear it by 1 mm. or less. If glass is used on

the windows of the inner case it is necessary to have a large hole closed by a plug in the round side of the case so that a pair of scissors with narrow blades can be inserted to cut the fibre.

Should the fibre when in use accidentally stick to the side of the case, it can always be set free, if the wall of the case is clean, by letting air very carefully into the electrometer and then giving the electrometer a violent to and fro rotatory motion about its axis, the plate being charged to about 60 V.

If difficulty is experienced in fitting a fibre which will move in the focal plane of the telescope, it may be necessary to substitute a narrow plate for the broad plate shown in figure 1. When this is done the plate-voltages necessary with a given fibre are higher and the calibration curves are not quite symmetrical, as in figure 2 where the plate used with fibre no. 1 was only 1/16 inch wide.

Earthing-keys using mercury or other liquids are not to be recommended. A dry clean brass-to-brass earthing key, totally enclosed in an earthed brass case and operating without friction, preferably by magnetic action, is satisfactory.

§ 8. ACKNOWLEDGMENTS

I am very grateful to Prof. Barkla for his supervision and for reading the paper, and to Mr Dymond for his interest in the work and in the paper. I acknowledge the benefit I have obtained from frequent discussions with Mr Stewart Brown, and I thank him for his collaboration in part of the work. I also thank Mr Mollison who has constructed some models of the electrometer for me according to the design given in this paper. The latter part of the work was carried out under the auspices of the Carnegie Trust for the Universities of Scotland.

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DISCUSSION

Mr R. S. Whipple. The range of sensitivity covered by the instrument is very great, and I should like to ask the author whether, in practice, he finds that he is able to hold the vacuum sufficiently constant to obtain a good sensitivity over any length of time? This is not important if the instrument is being used as a detector, but if it is being used for measuring then it is essential that the vacuum be maintained constant.

I was interested to see that the author has found that a heavily sputtered fibre was almost useless. This has been our experience with the Lindemann electrometer. At the suggestion of Mr T. C. Keeley, we now sputter the fibres carrying the moving system in the Lindemann electrometer with the minimum amount of metal possible. In this way a fibre, instead of the fibre plus the metal coating, is able to keep control, and as a result much more reliable zeros are obtained.

Prof. A. F. C. Pollard. In view of the several sources of energy-loss during motion of the fibre, I do not feel that the behaviour of the instrument has been fully investigated if the aperiodic hysteresis loop of instrumental variance, which would include with elastic hysteretic losses the unknown behaviour of the fibre end support, has not been plotted for the full range of the indications.

Mr J. Guild suggested that it might perhaps be possible to make a solid solution of metal in fused quartz and thus avoid any undesirable effects that might arise from a coating of sputtered metal on the fibre.

Mr W. H. White referred to the author's suggestion on p. 181 that it might become possible to see the Brownian movement with the naked eye, and pointed out that a fibre fine enough to give such large movements would itself be invisible.

Author's reply. It is important that the air pressure in the electrometer be constant, especially if the fibre is strongly illuminated and a high sensitivity is being used, because under these conditions the displacement of the fibre due to radiometer forces is relatively large and it varies with the air pressure. I find, however, when the brass case of the electrometer is properly vacuum-tight and when good low-vapour-pressure wax is used to seal the windows and insulators, that in spite of the small volume of the case the vacuum is well maintained. It is very advantageous of course to increase the volume of the electrometer by connecting it to a large airtight vessel. Sometimes it is possible to make measurements, even if the electrometer is not airtight, by running the pump continuously, an adjustable auxiliary leak being used if necessary.

I wish to emphasize, in reply to speakers who have referred to the hysteresis, the fact that the metal coating on the fibre can be made so thin that hysteresis effects entirely disappear. It is therefore unnecessary to investigate in detail the aperiodic hysteresis loop of instrumental variance, since we know already that with a good fibre it is negligibly small.

When very fine fibres are used it is easy to arrange the illumination so that the fibre is observed against a dark background by means of the light which it diffracts.

I hope to publish soon some measurements of the charge-sensitivity of the electrometer.

534.2.07

A MECHANICAL WAVE MODEL ILLUSTRATING ACOUSTIC AND ELECTRICAL PHENOMENA

By GILBERT D. WEST, D.Sc., F.Inst.P., Military College of Science

Received August 25, 1933. *Read December* 1, 1933.

ABSTRACT. A mechanical model is described which illustrates a band-pass filter. The model consists of a series of equal masses suspended on equal lengths of straightened watch-spring from a rigid bar. Through holes bored in the masses, which are equally spaced, is threaded a piece of elastic. One end is fixed, and the other can move with a simple harmonic motion, communicated by means of a rocker arm attached to a small motor. Wave-transmission along the system takes place only if the frequency falls within a certain range. Very high and very low frequencies are not transmitted.

The theory of the model is discussed, and various analogies to physical phenomena are pointed out.

§ 1. INTRODUCTION

The problem of the transmission of a disturbance through a medium with resonant properties has always been of fundamental importance in physics, but it has more recently gained special interest as the result of the invention of acoustic and electrical filters, and also of the study of the Heaviside layer and the electrodeless discharge. It is natural, under these circumstances, to turn for illustration to some mechanical model where the vibrations are made visible.

So far as transmission along a uniform string is concerned, there is available, of course, the classical method of Melde. Sir Ambrose Fleming* extended Melde's experiment by replacing the fork by an electric motor with a rocker arm, and was thus able to show, in addition, the transmission of waves along a string loaded with beads. The author does not know, however, of any further experimental extension of this method.

§ 2. DESCRIPTION OF THE MODEL

The present model consists of a series of equal masses suspended on equal lengths of straightened watch-spring, so that each unit has a definite natural period. Through holes bored in the masses is threaded a piece of elastic, such as is used for toy aeroplanes, and the whole system therefore gives the appearance indicated in figure 1.

The springs are clamped at the top by a rigid metal bar and are equally spaced. One end of the elastic runs over a pulley, and the other is attached to a rocker arm coupled to a small electric motor. The rocker consists of a light pivoted rod kept

* *Proc. Phys. Soc.* 26, 61 (1913). Note also in this connection the well-known model of Vinycomb (Griffin and Tatlock, Ltd.).

in contact with the edge of an eccentrically mounted brass disc by means of a spring. The rocker, when in action, thus communicates to the string, in a horizontal line at right angles to its length, an approximate simple harmonic motion. The speed of the motor is recorded by a tachometer and is controlled electrically by a potentiometer resistance, or mechanically by a continuously variable gearing of a simple kind. In this way, vibrations ranging from about 3 to 20 periods/sec. are obtainable.

§ 3. GENERAL DISCUSSION OF THE ACTION OF THE MODEL

When very slow vibrations are applied to one end of the string, it is noticed that the nearest masses vibrate, but that the amplitude of vibration rapidly decreases from mass to mass. Nothing of the nature of ordinary wave transmission takes place.

With increase of speed of the motor, however, a certain critical speed is reached when a sudden change sets in. Waves travel freely along the string, and may form stationary systems. Finally, when a high speed is reached, wave transmission ceases again, and the vibrations die out at a short distance from the spot where they are initiated. The refusal of the system to transmit high-speed oscillations is very striking.

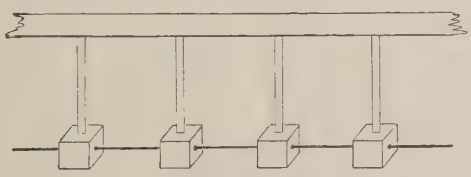

Figure 1.

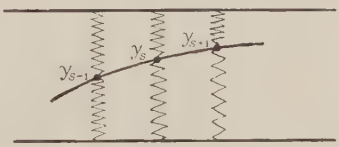

Figure 2.

In order to understand the interesting features that the model thus displays, it is necessary to consider the equations of motion.

Suppose the elastic is of unlimited length, and let T be its tension. Let m be the mass of each load, and a their distance apart. Let y_s be the displacement of the sth mass, and k^2 the restoring force due to the spring per unit displacement. (In figure 2 the springs are represented as spirals merely for the sake of the diagram.) T, m a, y_s k

We thus have

$$m\frac{d^2y_s}{dt^2} = T\left(\frac{y_{s+1}-y_s}{a}\right) - T\left(\frac{y_s-y_{s-1}}{a}\right) - k^2y_s \qquad \ldots\ldots(1),$$

and we require the solution of this equation subject to the condition that the first mass moves with a simple harmonic motion. If we decide further to consider only the steady state, then any other mass must also move with a simple harmonic motion of this same period.

Clearly there are three solutions that concern us, but the one that relates to the free transmission of waves is the simplest to begin with.

Let $$y_s = Y\cos(pt - qsa) \qquad \ldots\ldots(2)$$

be tried as a solution, where $p = 2\pi n$, n being the frequency, $q = 2\pi/\lambda$, λ being the wave-length, and where Y is the maximum displacement. p, n, q, λ Y

By substitution in (1) we find, on reduction,

$$-mp^2 = \frac{2T}{a}(\cos qa - 1) - k^2 \qquad \ldots\ldots(3).$$

Provided, therefore, that we satisfy this condition, (2) will be a solution. Rewriting (3) in terms of n and λ we have

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma}\sin^2\frac{\pi a}{\lambda} + \frac{k^2}{m}\right)}.$$

It is clear that n must be kept between the limits

$$\frac{k}{2\pi\sqrt{m}} \text{ and } \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}.$$

In the former case $\lambda = \infty$, and in the latter case $\lambda = 2a$. Within these limits therefore (2) is a solution.

Consider now the case when the vibrations imposed on the first mass are very slow. The apparatus shows, as has already been pointed out, that the amplitudes of vibrations of succeeding masses diminish rapidly, and it is perhaps natural therefore to try a solution of the type

$$y_s = Ye^{-K_1 sa}\cos pt \qquad \ldots\ldots(4),$$

where K_1 is a constant.

This on substitution in (1), and after reduction, gives

$$-mp^2 = \frac{2T}{a}(\cosh K_1 a - 1) - k^2,$$

or

$$\cosh K_1 a = 1 + \frac{ak^2}{2T} - \frac{4am\pi^2 n^2}{2T} \qquad \ldots\ldots(5).$$

Since the hyperbolic cosine of an angle varies from 1 to ∞, we can satisfy (5) provided $n \leqslant k/2\pi\sqrt{m}$, and (4) is therefore a solution for the range from this value of n, to $n = 0$.

It now remains to consider the range above the value

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}.$$

Here no obvious solution suggests itself. The wave dies out as before, but the possibility of combining an exponential and a cosine term has already been exhausted. It will be shown later that the model itself suggests the type of solution required, but admittedly it does this much more easily when we know the result in advance. I am indebted therefore to my mathematical colleague Mr Wright for directing me to, and amplifying, a valuable source of information* where a similar problem relating to pendulums connected with springs, together with the whole subject of linear difference equations, is discussed.

The modification to (4) necessary to make it apply to the highest frequencies is simple. In fact only a multiplication by $(-1)^s$ is necessary, so that we can now write

$$y_s = Y(-1)^s e^{-K_2 sa}\cos pt$$

* Bateman, *Partial Differential Equations of Mathematical Physics*, p. 233. See also a paper by T. B. Brown, *J. Opt. Soc. America*, **8**, 343 (1924).

and this, on substitution in (1), gives on reduction

$$\cosh K_2 a = \frac{4am\pi^2 n^2 - ak^2}{2T} - 1 \qquad \ldots\ldots(6).$$

Since again the hyperbolic cosine of an angle varies from 1 to ∞ we can satisfy (6) provided

$$(4am\pi^2 n^2 - ak^2)/2T \geqslant 2,$$

i.e. provided $n \geqslant \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}$, which is the range we want.

Hence we have three solutions:—

From $n = 0$ to $n = k/(2\pi\sqrt{m})$, $\quad y_s = Ye^{-K_1 sa}\cos pt$;

from $n = k/2\pi\sqrt{m}$ to $n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}$, $\quad y_s = Y\cos(pt - qsa)$;

from $n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}$ to $n = \infty$, $\quad y_s = Y(-1)^s e^{-K_2 sa}\cos pt$,

and the whole range is thus covered.

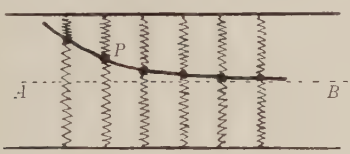

Figure 3.

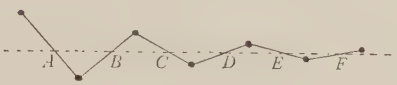

Figure 4.

These solutions represent satisfactorily the features the model exhibits, but it is possible to obtain much information in a general way, and thus further to interpret the results.

If a simple harmonic motion of frequency lower than the critical frequency is imposed on the end load, then, in the steady state, any load near by, say P, figure 3, must also vibrate with this frequency.

In order that this may happen, the restoring force must be less than that due to the spring alone and therefore the string must exert a force in opposition. Hence the string must always present a convex face to the central line AB, and thereby hold back the masses. No wave transmission, therefore, is possible. The solution $y_s = Ye^{-K_1 sa}\cos pt$ exhibits the exact shape of the convex curve from time to time.

Let the frequency of the impressed simple harmonic motion be now increased until it reaches that of one of the loads on a spring. Then again, in the steady state, P will vibrate with this frequency, but this will only be possible if there is no interference from the string itself. Hence all the masses must move backwards and forwards in phase, and the string will be straight. It could be said, in fact, that a wave of infinite length was being transmitted. For the transmission of a higher frequency, it is necessary that the restoring force due to the springs should be aided by the string itself. The latter must therefore, in the displaced position, always be concave to the central line AB. The curve $y_s = Y\cos(pt - qsa)$ achieves this.

As the frequency imposed on the end load increases, so does the concavity of

the string; in other words the waves become shorter and shorter. A further transition is reached, however, when the half wave-length is equal to the distance between the masses. Each mass now vibrates in such a way that the half-way points between it and the adjacent masses remain fixed. A mass can vibrate still faster, however, if the fixed points are arranged as in figure 4 at A, B, C, D, E, F, so that the mass is nearer to one fixed point than to the other. Here it is clear that the restoring force on each mass is greater than before, but the result is only secured by an ever diminishing amplitude. The solution

$$y_s = (-1)^s Ye^{-K_2 sa} \cos pt$$

is an analytical representation of figure 4.

Clear general ideas may thus be obtained of the working of the model.

§ 4. QUANTITATIVE EXPERIMENTS WITH THE MODEL

The region that lends itself most to experiment is the transmission band. The limit, on the high-frequency side, should be the frequency of a complete element

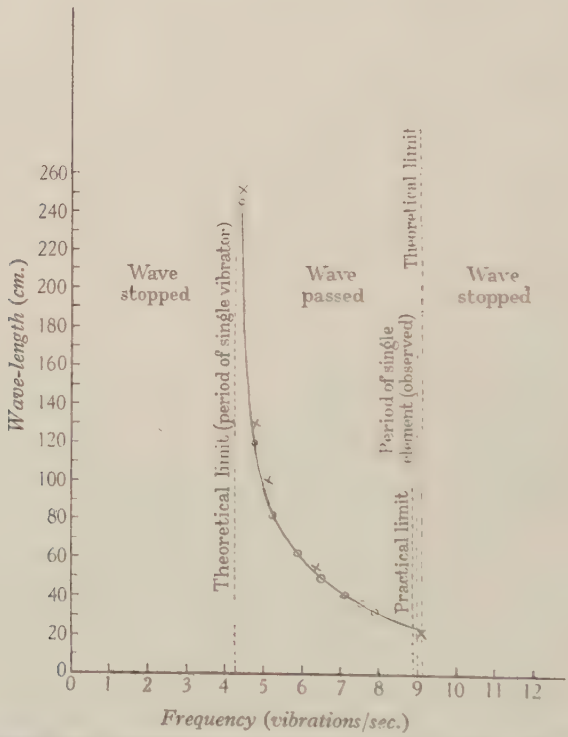

Figure 5. Band-pass filter. Average mass of vibrator, 14 gm.; average frequency of vibrator, 4·3 vibrations/sec.; distance apart of vibrators, 11·0 cm.; tension on elastic, 100 gm.

consisting of a mass and two lengths of string on either side equal to the semi-distance of separation of the masses. This frequency can be determined without difficulty and a comparison can be made with the frequencies the system will pass. An actual case is shown in figure 5.

Again, the lower limit should be the frequency of a single vibrator. This also is shown in figure 5.

In the range between, it is possible so to adjust the frequency of vibration that the half wave-length associated with it corresponds either with the length of the string or with some submultiple of this. In that case a system of stationary waves is formed, and there is no difficulty in obtaining several loops. By measuring these, and by measuring at the same time the speed of the motor, sufficient observations can be obtained to draw a graph relating wave-length and frequency. Such an experimental relationship is shown in figure 5, the observation being marked with circles.

It will be noticed that the data provide us with the means of calculating a relationship between wave-length and frequency from the formula

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma}\sin^2\frac{\pi a}{\lambda} + \frac{k^2}{m}\right)}.$$

The calculated values are marked with crosses.

Considering the various mechanical imperfections of the apparatus, the general agreement is satisfactory.

Since wave-velocity is given by the product of wave-length and frequency, it is easy to find this quantity from the graph. The wide range experimentally realizable is worth noting. Thus with a half wave-length equal to the length of the apparatus, the wave-velocity is 1080 cm./sec., whereas with a half wave-length equal to 16·3 cm. it is only 265 cm./sec.

§ 5. ANALOGOUS ACOUSTIC AND ELECTRICAL PHENOMENA

The most obvious analogy of the apparatus is to the acoustic and electrical filter circuits now well known*. In the electrical case the analogy is fairly complete, and the band-pass limits may be written down straight away. They are in fact the frequencies of the vibrator and the complete element. In the acoustic case, as Stewart found†, the inertia and elasticity factors are not so well located. Nevertheless the general analogy to the mechanical example remains, and only a band of frequencies is passed.

If (1) is reduced to differential form by making a very small, we have

$$m\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2} - k^2y \qquad \text{.....(7);}$$

m now refers to the mass per unit length, and k^2 to the force exerted by the springs per unit length per unit displacement.

This equation is now identical in form with that used by Sir J. J. Thomson to explain the properties of the Heaviside layer and the electrodeless discharge‡. It will be noticed that whereas the lower limit of transmission can be written $k/2\pi\sqrt{m}$, the upper limit has risen to an infinite value.

* Wood, *Sound*, p. 460 (1930).

† *Phys. Rev.* **25**, 90 (1925).

‡ *Proc. Phys. Soc.* **40**, 82 (1928).

We have therefore a high-pass filter. Waves below a certain frequency cannot penetrate such a medium, and, if incident upon it, are reflected back in the way now made well known by the extensive work of Appleton and others.

In the case of the electrodeless discharge, when the ionization is strong the frequency of the alternating currents is below the pass frequency, and the discharge consists of a ring close to the glass. The ring is very intense near the glass but fades away rapidly as we move inwards. In the case of the model, this is analogous to the way in which a vibration below the critical frequency fades away as we pass from mass to mass.

The model thus supplies a helpful analogy to a number of physical phenomena.

DISCUSSION

Dr W. G. Bickley. My attention was first drawn to the possibilities of mechanical models of wave-filters by Crandall's* detailed examination of a set of equidistant heavy beads on a stretched string. This model, as he shows, would behave as a low-pass filter. A point which writers on electrical filter circuits do not mention is that the cut-off frequency is the frequency of a normal mode for the infinite system, in which neighbouring elements execute similar oscillations. A mechanical system with two such normal modes might be expected to behave as a band-pass filter. I conceived a model in which a number of equal heavy bars were joined together and were supported by equal springs at the joints; on paper, this shows the same properties as Dr West's model. It also has one interesting feature, that by increasing the radius of gyration of an element the cut-off frequencies can be made equal, so that all frequencies are stopped.

Mr W. F. Floyd. It is evident, from the nature of the problem, that the most general solution is

$$y_s = Ye^{-K_1 sa}\cos(pt - \phi s) \qquad \ldots\ldots(4a),$$

ϕ where ϕ is the phase displacement per section of the filter. On substitution in (1) in the paper, (4*a*) gives

$$\cos\phi\cosh K_1 a = 1 + ak^2/2T - 4am\pi^2 n^2/2T \qquad \ldots\ldots(5a).$$

The above equations, (4*a*) and (5*a*), are analogous to (4) and (5) in the paper, but, since $+\infty < \cos\phi\cosh K_1 a < +\infty$, equation (5*a*) is valid for all values of n, whereas equation (5) is valid only when $k = 2\pi n/m > 0$. Free wave-transmission occurs when $K_1 = 0$; and at any instant it is evident that $\phi\lambda = 2\pi a$. Hence

$$n = \frac{1}{2\pi}\sqrt{\left(\frac{4T}{ma}\sin^2\frac{\pi a}{\lambda} + \frac{k^2}{m}\right)},$$

and $\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)} \geqslant 2\pi n \geqslant \sqrt{\frac{k^2}{m}}$, as in the paper.

When $2\pi n < \sqrt{\frac{k^2}{m}}$, $\cos\phi$ must be positive; when $2\pi n > \sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)}$, $\cos\phi$

* *Vibrating Systems and Sound* (Macmillan, 1927).

must be negative; when $2\pi n$ lies between these values, cos ϕ may have any value from + 1 to − 1.

In particular, at the critical value, when $\phi = 0$, and $2\pi n = \sqrt{(2T/ma + k^2/m)}$, the whole system moves without phase-difference between its elements.

It must be emphasized that so far no account has been taken of reflection from the fixed end of the system. For this a term $Y_1 e^{-K_1 sa} \cos(pt + \phi s + \theta)$ must be added to y_s in equation (4*a*). This added term is only important when $K_1 = 0$ and when $K_1 \to 0$. In the former case, $Y_1 = Y$ but the analysis becomes heavy.

In the electrical transmission-line analogue, the critical value, $\phi = 0$, corresponds to the case of optimum operation of the line, when no standing waves are present—i.e. there is a maximum transference of power over the line. Since a transmission line should never be operated so that wave-reflection occurs, the analysis for this case need not be considered. The model cannot illustrate the case of maximum power-transmission unless the remote end of the system be arranged to dissipate power into a characteristic load. For the range

$$\sqrt{\left(\frac{4T}{ma} + \frac{k^2}{m}\right)} \geqslant 2\pi n \geqslant \sqrt{\frac{k^2}{m}},$$

the model appears to be an exact analogue of a Lecher-wire system, but the reflection term should be added to (4*a*).

In the case of the electrical band-pass-filter analogue the band-width is generally small in comparison with the median frequency. On this assumption, the band-width is $2T\,ak\sqrt{m}$ in angular frequency measure; the condition cos $\phi = 0$ gives the median frequency.

Mr T. B. Vinycomb. Dr West's model shows the characteristics of a band-pass filter very clearly. Historically, the same type of wave model was exhibited to the British Association in 1877 at Plymouth. Instead of a spring suspension for the masses, strings were used, so that they formed a series of connected pendulums. The natural periods were much longer than in Dr West's model as the object was to demonstrate group velocity. According to the account in *Nature* this phenomenon was very clearly shown. No mention was made of the peculiarities of the propagation at high or low frequencies.

The solutions for the different bands can be obtained very simply as follows:

Put $4T/m = \omega_2^2$ and $k^2/m = \omega_1^2$; then when a motion with time factor $e^{\alpha\omega t}$ is being performed let there be a phase-difference ϕ between successive masses, so that

$$y_s = Ae^{j(\omega\alpha - n\phi)}.$$

Then we get from equation (1)

$$-\omega^2 = \frac{\omega_2^2}{2}\left(\frac{e^{j\phi} + e^{-j\phi}}{2} - 1\right) - \omega_1^2;$$

that is
$$\tfrac{1}{2}\omega_2^2(\cos\phi - 1) = \omega_1^2 - \omega^2,$$

or
$$\sin\tfrac{1}{2}\phi = \sqrt{(\omega^2 - \omega_1^2)}/\omega_2.$$

This can be satisfied by real values of ϕ in the region

$$\omega_1 < \omega < \sqrt{(\omega_1^2 + \omega_2^2)} \text{ only.}$$

α, β Write $$\phi = \alpha + j\beta,$$

and we get

$$\sin \tfrac{1}{2}\alpha \cosh \tfrac{1}{2}\beta + j \cos \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\beta = \sqrt{(\omega^2 - \omega_1^2)}/\omega_2.$$

(i) If $\omega < \omega_1$,

$$\cos \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\beta = \sqrt{(\omega_1^2 - \omega^2)}/\omega_2,$$
$$\sin \tfrac{1}{2}\alpha \cosh \tfrac{1}{2}\beta = 0.$$

Hence $\alpha = 0$, so that all the masses move in phase, and

$$\sinh \tfrac{1}{2}\beta = \sqrt{(\omega_1^2 - \omega^2)}/\omega_2,$$

where $e^{-\beta}$ is the ratio of successive amplitudes. β decreases from $\omega_1\ \omega_2$ to zero as ω increases from 0 to ω_1.

(ii) $\omega_1 < \omega < \sqrt{(\omega_1^2 + \omega_2^2)^2}$,

$$\cos \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\beta = 0,$$
$$\sin \tfrac{1}{2}\alpha \cosh \tfrac{1}{2}\beta = \sqrt{(\omega^2 - \omega_1^2)}/\omega_2.$$

Thus $\beta = 0$; there is no attenuation but normal wave propagation. α ranges from
d 0 to π. The wave velocity is equal to $\omega d\ \alpha$ and the group velocity is $d.d\omega\ d\alpha$ if d is the distance between the suspended masses.

(iii) $\omega \geqslant \sqrt{(\omega_1^2 + \omega_2^2)}/\omega_2$,

$$\sin \tfrac{1}{2}\alpha \cosh \tfrac{1}{2}\beta = \sqrt{(\omega_2^2 - \omega_1^2)}/\omega_2 > 1,$$
$$\cos \tfrac{1}{2}\alpha \sinh \tfrac{1}{2}\beta = 0.$$

Here α remains equal to π, and successive masses move in opposite phases. β increases from 0 without limit as ω increases. This is the ordinary solution of a band-pass filter. In using any analogy between a mechanical or acoustic filter and an electrical filter care must be taken in identifying corresponding parts. For instance, in this model the elastic string connecting the masses in series is the analogue of capacitances shunting the line in the electrical case, and the spring strips which in the model appear in shunt, as it were, correspond to capacitances in series in the electrical line. It is best to work out each case separately, and compare the equations obtained in order to find the corresponding parts. The limiting frequencies are always found to be natural frequencies of the structure. Dr West's discussion of the motions of the model brings this out very clearly.

Dr L. E. C. Hughes. In an electrical filter, the transmission engineer endeavours to arrange the elements so that there is no reflection of wave-energy in the pass band of the filter. For a complete representation of an electrical filter it is therefore necessary to use a termination which has the correct impedance over the pass range. In electrical filters this is possible in most practical cases, but would appear to be difficult in a mechanical model, although the cut-off frequencies can be clearly demonstrated. If in a T section of an electrical filter the total series impedance is Z_1 and the shunt impedance is Z_2, the cut-off frequencies are given when $0 < Z_1/4Z_2 < 1$. Within this band there is maximum transmission, while outside

there is attenuation. To find the cut-off frequencies it is only necessary to insert ω in the above expressions for the impedances and solve for ω.

Capt. C. W. Hume. Mechanical wave-filters have been the subject of a number of inventions; the following British patent specifications may be of interest: 268,054; 258,395; 248,900; 243,321; and 221,792. Since it is a common practice to use only a small number of sections, the effect of terminal impedance is all-important. Within the transmission band the characteristic impedance is theoretically a pure resistance; could not the author illustrate this aspect of the problem by fastening the end B of his elastic to some sort of dash-pot having a minimum of inertia and resilience? At present his terminal impedance is infinite.

Author's reply. I have followed up the suggestion made by Capt. Hume that the impedance of the terminal end of the model should be reduced, and, accordingly, light aluminium vanes have been attached to the end loads. By this method considerable air-damping is obtained and the model demonstrates, of course, that power is transmitted, but so far I have not been able to give any very striking demonstration of the variation of power transmission with frequency.

I agree with Dr Bickley that insufficient attention is given to the relation of cut-off frequencies to the free periods of the vibrating elements. In simple electrical filters these frequencies can be written down from direct analogy with the present model.

Mr Floyd's treatment of the equations is mathematically more comprehensive than my own, but the physical distinctness of the three cases makes me still prefer three direct solutions.

Mr Vinycomb's elegant use of imaginary quantities brings out well the continuity of algebraic form. The model he mentions would not appear to be well adapted for showing band-pass properties, but I have succeeded in modifying the model that he himself invented for this purpose.

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THE MEASUREMENT OF IMPACT STRESSES IN CONCRETE

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ABSTRACT. A quartz piezo-electric gauge, used with a cathode-ray oscillograph for recording, to measure impact stresses in concrete has been developed. It is being employed to study the stresses in driven reinforced-concrete piles. Records illustrating the character of these stresses are given.

§ 1. INTRODUCTION

The piezo-electric response of a quartz crystal to applied loads is practically instantaneous and a gauge making use of this property, with a suitable oscillograph for recording, forms the most accurate instrument yet available for studying impulsive or rapidly fluctuating pressures. The method has recently been applied to a number of problems. Gas pressures in the breech of a gun during the explosion of a charge have been measured by Webster[1]; lathe cutting pressures, the starting torque of an electric motor, and impact stresses in a steel wire by Kluge and Linckh[2]; Grebel[3] has described its use for recording the pressures in internal-combustion motors; Gechter and Laird[4] have used apparatus similar to that of Kluge and Linckh in an investigation of the three components of tool pressure in a lathe.

The present paper is concerned with a further application—the investigation of impact stresses in concrete—and describes apparatus developed for the investigation of stresses occurring in reinforced concrete piles during driving. The work is being carried out at the Building Research Station in co-operation with the Federation of Civil Engineering Contractors.

§ 2. PRINCIPLES OF THE METHOD

The method is based on the fact that a hemihedric crystal with inclined faces, such as quartz, when compressed or elongated along one of the hemihedral axes, develops, at particular regions of the crystal, electric charges which are proportional to the applied force. The crystals forming the sensitive elements of gauges are cut from the natural material in the direction shown in figure 1 (*a*), that is, with a pair of faces perpendicular to the electric axis. Two ways of using the crystals piezo-electrically, which differ in the direction of application of the pressure, have been employed in previous researches, and the shape of the finished piece depends on which of these is used. In the first, figure 1 (*b*), in which the pressure is applied

along the electric axis, the quartz is generally cut in the form of a circular plate whose flat faces carry electrodes and are perpendicular to that axis. Two crystals are usually placed back to back so that the two outer electrodes are of the same polarity and are joined together and earthed. The middle electrode of opposite polarity is insulated by the quartz. In the second method it is most convenient to use a cube or long rectangular prism whose faces lie perpendicular to the electric, optic, and third or neutral axis. The pressure is applied along the third axis so that the electrodes, again on the faces perpendicular to the electric axis, are not subjected to compression, and for this reason the second method is adopted in the present research. The crystal, cut as in figure 1 (*c*), is mounted in a small water-tight chamber, described later in detail, which is cast in the concrete pile. Leads are brought out to a standard condenser across which, when the pile is stressed, a voltage is set up.

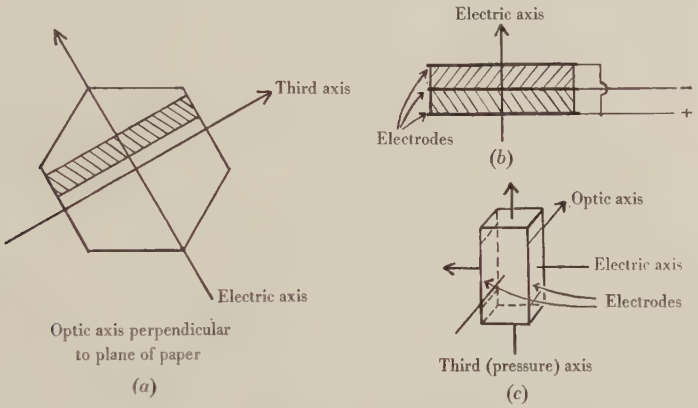

Figure 1. Position of axes in cut quartz crystals.

The two other main parts of the apparatus are the amplifier and the cathode-ray oscillograph. The amplifier is an important, and with commercial oscillographs an essential, component. Besides the requisite magnification of the voltage across the standard condenser, it provides, by the use of a special input valve, the very high resistance load across the quartz, which is necessary to reduce leakage to a negligible amount. This is particularly important for calibration, which is done by static loading. The input valve is, in the amplifier here described, an electrometer triode, but the double-grid valve has been used with equal success by Kluge and Linckh and other workers. The development of such valves of extremely high grid-filament resistance has rendered possible the use of the piezo-electric method with commercial cathode-ray oscillographs and with mechanical oscillographs of the Duddell type.

The cathode-ray oscillograph is unequalled for the conversion of transient electrical impulses into optical deflections. Its characteristics allow full advantage to be taken of the faithfulness of response of the crystal to pressure changes, and modern commercial examples have the additional merits of ease and dependability of working. Figure 2 is a theoretical diagram of the gauge, amplifier and oscillograph with auxiliary apparatus.

§3. EXPERIMENTAL

The gauge. Constructional details are given in figure 3. The quartz crystal 1 is a rectangular prism $\frac{3}{4} \times \frac{1}{4} \times \frac{1}{4}$ in.^{3} cut as in figure 1 (*c*) so that one pair of faces is perpendicular to the electric axis, and a second pair to the optic axis. Those perpendicular to the electric axis carry the electrodes 2 of electrolytically deposited copper. The electrodes cover about three-quarters of each face, leaving at each end a short length of quartz for insulation.

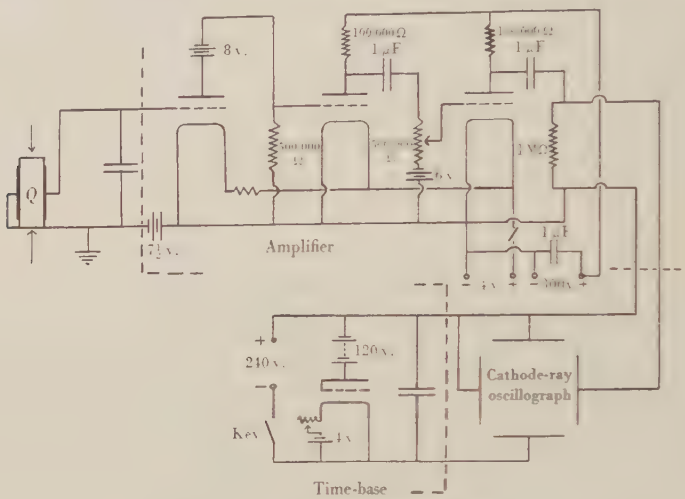

Figure 2. Electrical connections.

The prism is held between steel plates 3 to which the thrust is applied through steel balls. At one end a stiff spring 4 is inserted between ball and plate to reduce the stress in the quartz.

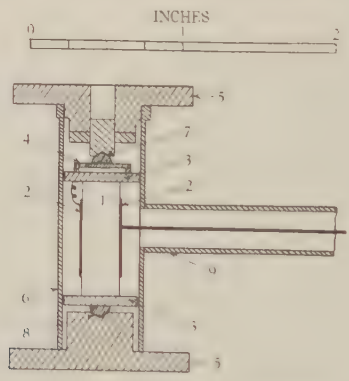

Figure 3. Section of a gauge.

Two heavy circular steel end-plates with projecting flanges 5, connected by a thin brass cylinder 6, form a water-tight chamber enclosing the crystal. The latter is held between conical seatings 7 and 8, of which one is adjustable. A highly insulated lead is brought out from one electrode through the brass tube 9, while the other electrode is earthed to the case.

When the apparatus is assembled a permanent load is put on the crystal by screwing up the adjustable seating 7. The gauge then responds to tension, which its flanged shape, affording a key for the concrete, enables it to follow.

The amplifier. The essential feature of an amplifier for piezo-electric work is that the grid-filament resistance of the input stage shall be very high. This is ensured by the use of the General Electric Company's electrometer valve which has a grid-filament resistance of $10^{15}\ \Omega$. The complete amplifier, figure 2, consists of this valve followed by two L.S. 5. B. valves. The first and second stages are direct coupled. Resistance-capacity coupling links the second to the third valve, and the third to the vertical deflecting plates of the oscillograph. The overall amplification is 48·7, to which the first stage contributes a factor of about unity. Provision is made for varying the amplification by means of a calibrated grid potentiometer in the second stage, but this control is rarely used as it is generally more convenient to vary sensitivity by altering the parallel capacity across the gauge. The amplifier is housed in a shielded box, into which the lead to the input grid is carried by a heavy ebonite tube.

The condenser in parallel with the gauge has a value of 0·001 μF. to 0·003 μF., large enough for small changes in capacity of leads to be negligible.

For the static calibration of gauges, the amplifier is arranged to have an input resistance of not less than $10^{12}\ \Omega$., but for working purposes this is with advantage reduced to about $10^{9}\ \Omega$., to allow charges induced on the grid by external disturbances to leak away in a reasonable time.

Recording-equipment. The oscillograph is manufactured by Messrs Cossor, Ltd., and is operated for photography at an anode potential of 1500 V. provided by a high tension unit from the a.-c. mains.

Records are taken with a stationary plate camera and a single-traverse time base. In this method[5], which is suitable for the registration of phenomena of short duration, a voltage varying linearly with time is impressed on the horizontal plates of the oscillograph at the appropriate moment so as to cause the spot on the fluorescent screen to move horizontally across, while the voltage from the gauge is simultaneously impressed on the vertical deflecting plates. The (stress, time) pattern thus delineated on the screen is photographed with a stationary camera.

The time-base circuit, figure 2, is made up of a condenser charged by a battery and discharged through a saturated diode by the opening of a switch at the correct moment. The speed of the spot across the screen is controllable between 1 and 10 m./sec. by variation of the condenser capacity and the saturation current of the valve.

A quarter-plate camera with a Ross Xpres f. 1·9 lens of 3 in. focal length is fixed relative to the oscillograph and gives a slightly reduced picture of the screen. A lens cap, hand operated, serves as shutter.

A time scale of milliseconds is impressed on the same plate as the record by a second exposure. After a stress record has been made the oscillograph is switched over to a fixed frequency 1000-cycle oscillator and a photograph of the wave-form is taken with the same time-base speed as before.

§ 4. CALIBRATION

Two operations are required for calibration. The overall sensitivity of amplifier and oscillograph is determined in the first and expressed as photographic deflection in mm. per grid volt of the amplifier. This is found by applying known voltages suddenly to the grid of the input valve and photographing the oscillograph deflections. Two switches are inter-connected so that the time base is released and the calibrating voltage is applied immediately afterwards. This gives a record of stepped type well adapted for measurement. The overall calibration was checked frequently during the taking of records in order that allowance might be made for any alteration in the sensitivity. No appreciable differences were observed.

The second part of the calibration is the determination of the sensitivity of the gauge in concrete in terms of charge developed per unit strain. The gauge is cast at the mid-point of the axis of a block of concrete 16 × 4 × 4 in.^{3} of the same constitution as the concrete in which stresses are to be measured, and subjected to the

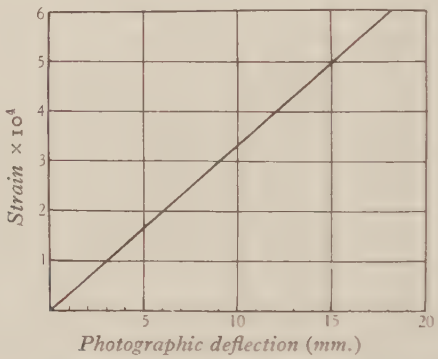

Figure 4. Sensitivity of gauge J. with 0·002 μF. condenser.

same treatment. The block is stressed up to 3000 lb. in.^{2} in a testing machine, measurements of strain being taken with a two-inch roller extensometer with telescope and scale, over the two-inch central portion in which the gauge is situated. The charge developed by the gauge produces across a standard condenser a voltage which is measured with a direct-coupled amplifier, constructed by removing the third valve of the amplifier and inserting a galvanometer with balancing circuit in the plate circuit of the second valve. Each time that it was used this instrument was calibrated by the application of known voltages from a potentiometer.

The procedure was to measure strain across the section containing the gauge, and charge developed for a number of increments of stress. A curve was then drawn between charge, or voltage across a standard condenser, and strain. Both parts of the calibration provide linear relationships, so that the resulting graph of photographic deflection against strain is a straight line, figure 4. The sensitivity in tension is obtained by extrapolation. In the interpretation of records, strains are converted to stresses by multiplying by the statically determined Young's modulus. This procedure is discussed below.

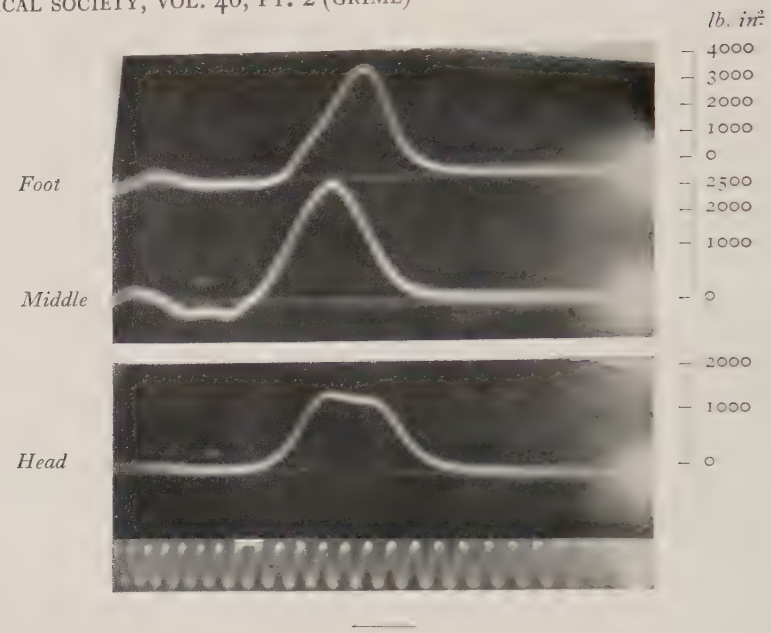

Figure 5. 12-inch drop of 1000-lb. hammer. Foot of pile bolted down to rigid base. Head buffered with 1 inch of felt.

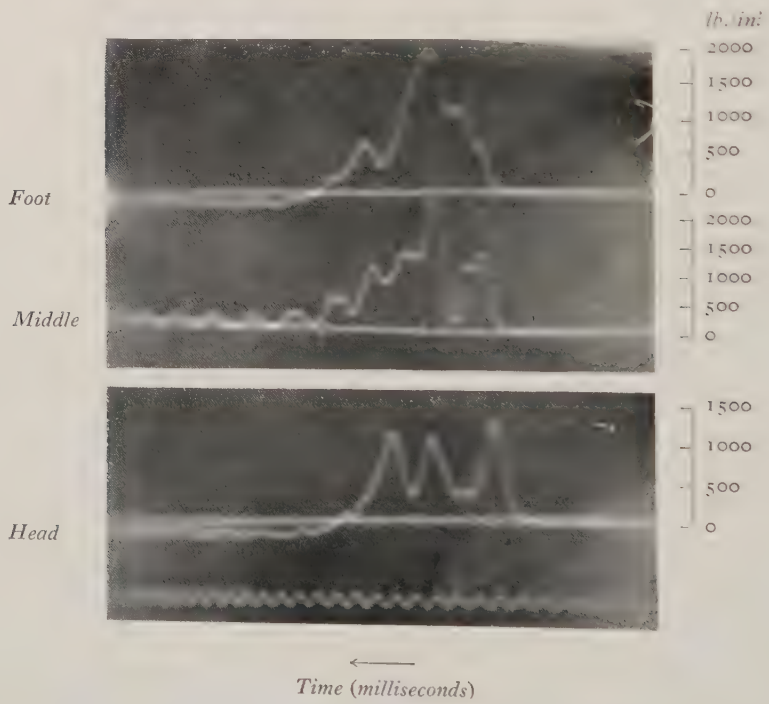

Figure 6. 6-inch drop of 1000-lb. hammer. Foot of pile on ½-inch felt on rigid base. Head buffer—½-inch felt beneath 3-cwt. helmet above which is a 12-inch elm dolly.

To face page 201

The accuracy of calibration in terms of strain is estimated to be subject to an error of ± 5 per cent. It is, however, possible to be in error by approximately ±10 per cent in the stresses calculated from these strains. The chief cause is probably the difficulty of assigning the correct modulus to the block of concrete in which the gauge is cast. Large differences of Young's modulus are frequently observed between blocks of concrete of the same mix and curing-treatment, and the only way of avoiding error is to measure the modulus directly over each gauge, which in the majority of cases is impracticable. There is also no direct evidence that the (stress, strain) relationship of concrete is the same in impact as that determined statically.

Two checks on the accuracy of stress-measurement may be made from the records. The first is the calculation of Young's modulus for a pile from the observed frequency of longitudinal vibration, and the second is the comparison of two calculations of the momentum of the hammer, the one made from the stress record at the head of the pile and the other from the measured heights of drop and rebound. These give the figure of ± 10 per cent and are further referred to in the next section.

§ 5. WORKING DETAILS. IMPACT STRESSES IN DRIVEN REINFORCED-CONCRETE PILES

Measurements of stress have been made in reinforced-concrete piles measuring 15 ft. × 7 in. × 7 in., in which gauges were cast 4 in. from each end and at the mid-point of the length. The piles were driven with a drop hammer falling freely.

The recording apparatus was set up in a hut measuring 10 ft. × 8 ft. close to the pile-driving frame. It was found that with the comparatively large parallel capacity used, generally 0·002 μF., unshielded leads of bare copper wire on porcelain insulators were perfectly satisfactory except on very wet days. A separate insulated lead from each gauge was brought to a change-over switch in the hut by which each could be connected in turn to the amplifier. The chambers enclosing the crystals were connected to a common earth lead and to the amplifier.

The switch in the time-base circuit was set up on the pile-driving frame on an adjustable mount and so arranged that the hammer in its descent tripped it at the correct instant for recording purposes.

A traverse speed of about 3·5 m./sec. across the screen was normally used. Since records from the three gauges in the pile could not be made simultaneously, a complete series for any one condition was obtained in three or, if a check was necessary, four consecutive blows.

Records for two different conditions of driving are reproduced in figures 5 and 6. Figure 5 is of the simplest type obtained. The pile is bolted down against a rigid concrete foundation and on the head a buffer, consisting of 1 in. of felt, is placed. The hammer weighs 1000 lb. and the drop is 12 in. Under these conditions the curves at head, middle and foot are smooth, and the maximum compression at the foot is approximately twice that at the head. After the hammer has rebounded, tensions are set up at the middle, and to a smaller extent at the foot. Figure 6 represents easier driving, the foot of the pile resting on a felt packing on the rigid base. The buffer is the arrangement used in practice, a heavy iron helmet in the top

of which is fitted an elm block or dolly, while beneath it a felt packing protects the pile-head. Mathematical investigation[6] leads us to expect that the pile will behave as a long thin rod, and that stresses in the pile will, at a given instant, vary along its length, and at any one point will fluctuate in a periodic manner. Quantitative agreement with theory has not yet been reached, but the records in figure 6 clearly illustrate the wave character of the stresses. The frequency of longitudinal vibration is well brought out in the record at the centre, where slight tensions are set up by the residual vibration. The apparent tensions at head and foot are due to electrical leakage. After the hammer has rebounded, the ends of the pile are antinodes and the middle a node. The frequency of vibration is 455 c. sec., whence Young's modulus for the pile is $5{\cdot}7 \cdot 10^6$ lb. in^2. The value calculated from the moduli of concrete and steel is $5{\cdot}35 \times 10^6$ lb. in^2. The (stress, strain) relationship from the impact record therefore agrees with that determined statically within 7 per cent.

This is the first check on the accuracy of measurement of stress, mentioned in the preceding section. The second is provided by calculating the momentum change of the hammer.

At the head of the pile the force exerted downwards over the cross-section is, at any instant, equal and opposite to that decelerating the hammer, provided that the effect of the buffer can be neglected. Then it is easily shown that the total momentum change

$$M(V - v) = A \int_0^T P.dt,$$

where M is the mass of the hammer, V and v its velocities immediately before and after striking, A the area of the pile-head, P the stress at the head of the pile, and T the duration of the blow.

The integral is proportional to the area under the (stress, time) curve of the gauge at the head, and the left-hand term is immediately determined from the heights of drop and rebound. Calculations thus made agree within 10 per cent.

§ 6. ACKNOWLEDGMENTS

The author wishes to acknowledge his indebtedness to Mr Wilsdon and Mr A. F. Dufton under whose supervision the work has proceeded; to Dr W. Glanville for advice and criticism, and to Mr C. T. Webster for general assistance. He desires also to express his grateful thanks to Mr J. F. Herd of the Radio Research Station for his generous assistance in the design and construction of the amplifier, time base, and oscillator.

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DISCUSSION

Mr H. W. Heath. The paper calls for criticism of the mechanical principle involved, namely, embedding in a concrete pile a foreign body (the gauge), obtaining data from the deformation of that body while so embedded, and interpreting the result as stress in the concrete.

The results would be completely misleading unless the Young's modulus for the complete gauge is the same as that of the concrete. The method of calibrating the gauge is subject to the same criticism.

The state of stress inside the test piece can be followed by considering a compound bar made by welding together, side by side, a copper and a steel bar, and applying an end load. The strain would be the same for both bars, but the stresses are in the proportion of the respective moduli; and the part of the load taken by either bar is the product of the area of that bar and the stress in that bar, the sum of these two products being equal to the total load.

Precisely the same conditions apply to the calibration of the gauge when embedded in the test block to determine the relation between strain and pressure (or charge). At the cross-section containing the top of the gauge the stress is not uniform, and the charge developed depends on the two moduli (concrete and gauge separately), and on the relative areas of test piece section and top plate of gauge. Moreover, the straight-line law obtained from the test block is not applicable to the reinforced pile, since the area of the pile is 49 in.^{2}, and about 2 in.^{2} of this is steel equivalent to, say, 30 in.^{2} of concrete, so that the steel rods take nearly half the load.

The essential information required for the test is the Young's modulus for the complete gauge, as shown in figure 3, when it is not embedded in concrete. A relation can then be deduced between the pressure on the end plates when embedded and the strain on the face of the test piece. This relation must then be modified because of the difference in area between pile and test piece, and further modified to allow for the share of the reinforcement in the load, and for the suddenness of the load applied to the pile as distinguished from the load applied statically to the test piece.

Bearing in mind that stresses in a cross-section can vary enormously at two points separated by only a centimetre, it would be miraculous if the stresses recorded were anywhere near those actually developed in the pile.

The only reliable information emerging from the test is the frequency of the longitudinal oscillations, and hence the Young's modulus for the whole pile.

Author's reply. Mr Heath has lost sight of the fact that the charge developed depends directly upon the stress in the quartz but only indirectly upon the pressure exerted upon the end plates when the gauge is in position. The strain suffered by the gauge is likely to be the same as that of the concrete in which it is embedded. Assuming this, it is readily seen that since the charge developed is proportional to the stress in the quartz, it will be proportional to the relative movement of the end plates and so to the strain in the concrete. We therefore measure strain in the

concrete, from which stress is derived in the usual manner. It is therefore not at all necessary for the interpretation of results to know anything of the constants of the gauge. An estimate of the stiffness is useful however for design purposes in order to ensure that the stiffness shall be approximately equal to that of the concrete.

Finally, we are at present interested only in the average stress over the cross-section, and not in the variation which must be present in a non-homogeneous material such as concrete. That this is being measured with reasonable accuracy is confirmed by the agreement now being obtained between the measured stresses and those calculated from theory.

535·247·4

AN AUTOMATIC PHOTO-ELECTRIC PHOTOMETER

By ERIC B. MOSS, B.Sc. (Lond.)

Received October 14, 1933. *Read December* 1, 1933, *with demonstration.*

ABSTRACT. A precision photo-electric photometer based on principles capable of wide application and operated from a.-c. mains is described. It is a flicker instrument but the simple shutter is on the spindle of a synchronous motor driven from the same supply as an a.-c. valve bridge. This is connected to an emission type photo-cell, and gives a directional output which automatically moves the neutral density wedge to the position of balance which is shown by a pointer. The wedge position is controlled electrically, being mounted on a galvanometer movement devoid of mechanical control.

A description of a densitometer for liquids is given.

§ 1. INTRODUCTION

NUMEROUS photo-electric photometers have been described whose general purpose is to compare two intensities of either transmitted or reflected light*. These instruments are as varied in design as in purpose, and usually they have been intended for laboratory use, so that to make one instrument suitable for general use it is necessary to design yet another apparatus.

Briefly, the object of the design is a reliable instrument which can be used by routine operators, or experimenters skilled only in their particular branch of science. In their hands it must yield, for subsequent interpretation, readings of optical density or reflective power with sufficient accuracy and in a minimum of time. The demand for reliability makes it necessary that the instrument shall work from alternating-current mains.

The alternative-path or flicker principle is used because it eliminates the effect of changes in the sensitivity of the photo-cell, or in the brilliance of the light source, but the optical system is so arranged as to overcome the difficulty sometimes experienced in adjusting the shutter so that one beam may be opened and the other closed at exactly the same rate. The design of the shutter also ensures that any flicker shall be of sinusoidal wave-form and thus be capable of being amplified efficiently by means of a simple valve amplifier. The valve amplifier gives a directional d.-c. output when the intensities are unequal. This is secured without the use of a commutator, since it is found in practice that a small commutator used in conjunction with a galvanometer can give trouble. Inequalities of the beams are balanced by means of a neutral optical wedge. This is moved automatically without relays, by the output from the amplifier, and its position is shown by a pointer on a scale. This feature has points of similarity with one described by A. C. Hardy†, but in the present instrument the end is achieved in a simpler manner.

* See the end of the paper.

† A. C. Hardy, *J. Opt. Soc. Amer.* **12**, 96 (1924).

§ 2. THE OPTICAL SYSTEM

A diagrammatic arrangement suitable for measuring optical densities is illustrated in figure 1. It has two symmetrical halves whose axes have a common origin at the lamp filament and terminate on the cathode of the photo-cell. One axis is through the lenses A_1, B_1, C_1, D_1 and prisms K_1, L_1, while the other passes through the components A_2, B_2, C_2, D_2 and L_2. The axes therefore intersect at O. An image of the lamp-filament is formed by lens A_1 on B_1, and the latter is calculated to produce an image of A_1 at the intersection O. The field lens C_1 forms an image of the

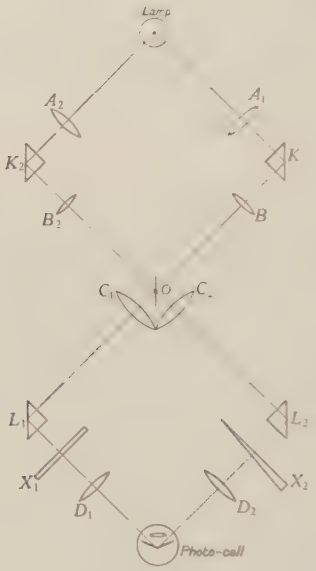

Figure 1.

filament at X_1, at which point is placed the unknown density. Subsequently the diverging beam is concentrated by lens D_1 on to the photo-cell cathode. The other half of the optical system is identical except for the insertion of the adjustable balancing wedge at X_2. Mounted at O, so as to be perpendicular to the plane containing the two optical axes, is an extension of an electric-motor spindle. This carries a thin disc slightly smaller in diameter than the image of A_1 or A_2, so that as the spindle rotates first one and then the other beam is obstructed. The quantity of light in each path will vary harmonically with time at a frequency double that of the motor-speed, and one will lead the other by 180°. This angular displacement is dependent on the lay-out of the optical components. It can therefore be made with precision and is not liable to derangement. Clearly, then, any inequality in the two beams will give rise to a sinusoidal light-fluctuation whose phase will depend on which beam is the brighter.

It will be seen later that only that harmonic portion of the photo-electric current

which is synchronous with the shutter rotation affects the amplifier. Therefore it is essential that this shutter shall rotate in a uniform field. It does not matter if the field is considerably larger than the circle of the disc, but the need for maximum sensitivity makes a large margin undesirable. In practice the diameter of the shutter is about 1 mm. less than that of the images of A_1 and A_2.

With a little modification the system can be made suitable for comparing the reflective powers of matt materials such as paper, cloth and powders. The standard and sample may be placed side by side just in front of the position occupied by the photo-cell in figure 1. In this position one beam would fall on each, and the photo-cell would be placed on the centre line of the system but facing the specimens so as to receive an equal proportion of the light reflected from each. The efficiency of such a simple system could be increased, for example, by the interposition of a field lens between specimens and photo-cell.

§ 3. THE VALVE AMPLIFIER

A phase-sensitive detector is necessary to take advantage of the phase-change which results from inequality of the two light-intensities, and the valve bridge due to C. R. Cosens* has been chosen as being very suitable. By its use rectifiers and smoothing accessories are dispensed with and this saves considerable space.

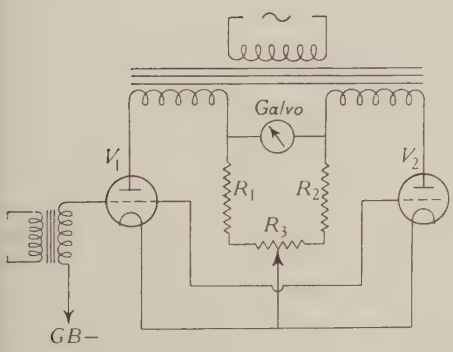

Figure 2*a*.

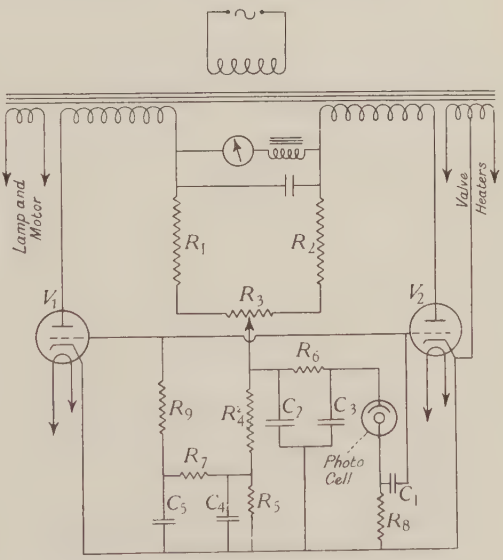

Figure 2*b*.

The simple circuit is shown in figure 2*a*. It consists of two valves V_1 and V_2 with their anodes connected to separate windings on the same transformer so that they are in anti-phase, the anode-cathode circuits being completed by the resistances R_1, R_2 and a potentiometer R_3. If the valves are matched and their grids

* British Patent Specification, No. 385,982.

tied together to the correct bias, then during one complete cycle each will pass the same current. This results in the mean current through R_1 being equal to that through R_2. The galvanometer can be zeroized even if the valves are not perfectly matched, by adjusting the tapping on the potentiometer R_3.

Suppose now that through the grid transformer a small alternating e.m.f. is applied to the valve grids in phase with the anode supply to V_1. During one half-cycle the grids will be slightly more positive than the steady grid-bias when the anode of V_1 is positive, and more current will flow through V_1 and R_1. During this half-cycle the anode V_2 is negative and no current passes through the valve. During the next half-cycle the grids are more negative and therefore less current will pass through V_2 and R_2 than passed through V_1 and R_1 in the previous half-cycle. The continuous effect is to impart a steady deflection to the galvanometer. Should the phase of the grids be reversed, then the galvanometer will be deflected in the opposite way because the mean current through R_2 will be greater than that through R_1.

Certain modifications are necessary to the valve bridge so that it may be suitable for operation entirely from alternating supply. The complete circuit is shown in figure 2*b*. A smoothing circuit has been added to the galvanometer, and further additions are resistances R_4 and R_5. The sum of these two is such that the total rectified current through the valves gives a potential-drop suitable for the operation of a photo-cell. The resistance R_5 is that for automatic grid-bias, and R_6 and R_7 are added in conjunction with a condenser for smoothing. The values of the resistances and capacitances used in the densitometer described later in this paper are as follows:

	Ω.		μF.	
R_1, R_2	3000	C_1	0·1	
R_3	400	C_2	2·0	(electrolytic)
R_4	20,000	C_3, C_5	0·5	
R_5	800	C_4	50·0	(electrolytic)
	MΩ.			
R_6, R_7	0·25			
R_8, R_9	2·0			

The transformer was wound to give 230 volts on the valve anodes. Valves of the Osram Catkin type were chosen partly on account of their small size. These have an impedance of 11,100 Ω. and a mutual conductance of 3·6 mA. V. The resistance of the moving coil of the wedge movement is 1000 Ω. and that of its choke 100 Ω.

Up to this point, by the galvanometer has been meant an ordinary instrument with a suspended or pivoted movement under a definite mechanical control. Hereafter this conception must be forgotten, as the movement used in the instrument is devoid of mechanical control and is held in its correct position electrically.

One requirement which this bridge satisfies is that it shall not be effected by even harmonics of the fundamental frequency, e.g. the frequency of the supply to the transformer. This is vital because the illumination from a gas-filled-lamp filament is modulated to a considerable depth at these multiple frequencies. The

second harmonic is predominant and it is essential to ensure that the voltage developed on the valve grids at this frequency shall not be sufficient to cause over-loading.

There may be conditions, as in spectrophotometry or in measurements with reflected light, when such intense illumination of the photo-cell may not be available. Under these circumstances the sensitivity can be increased by preceding the valve bridge with one stage of amplification. If this is done, however, it is more important than ever to ensure that over-loading of the bridge shall not be caused by the second harmonic.

An alternative to the use of this valve bridge would be to employ a straight amplifier with a dynamometer connected to the output and used as a detector. The fixed coils of the dynamometer would be fed from the supply, and the directional characteristic would be retained.

§ 4. THE MOTOR

The frequency co-ordination between the valve-bridge supply and the shutter-disc rotation is secured by the use of a synchronous motor connected to a winding on the same transformer. This motor is a four-pole machine and was specially designed for the instrument. The rotor turns at half the synchronous speed, but since one complete cycle of light-change takes place in half a turn the change is at the fundamental frequency. The motor is illustrated in figure 3.

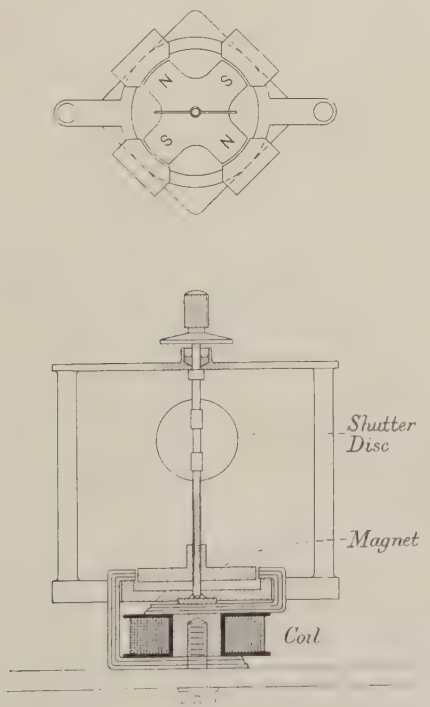

Figure 3.

It is further necessary that the alternating voltage which appears on the valve grids in the out-of-balance condition of the beams shall be in phase with the anode-voltage of the correct valve. If it is not so, clearly the sensitivity is reduced in the proportion of the cosine of the angle of error, provided that the latter is not greater than 90°. Should this be exceeded it means that the out-of-balance e.m.f. is coming into phase with the other valve supply, and this is equivalent to reversing the output of the bridge. The effect which this has will be mentioned later.

The phase is dependent on the position of the motor stator in relation to the optical work, and this is readily adjusted by rotation since the motor is held down by one central screw. At the base is the field coil which is wound for a 6-volt supply as it is connected in parallel with the lamp. Above this are the four poles, their magnetic path inside the coil being completed by a slotted mild-steel core. The poles are fastened to a brass disc which has a hardened steel bearing at its centre to support the vertical spindle on to which the cross-shaped permanent magnet is forced. Above it is the top bearing in a bridge piece, while higher up the shaft is the shutter disc and the starting head. This latter is used to start the motor by means of a sharp twist given with the finger and thumb. Synchronism is obtained on any commercial supply frequency, and such changes as are likely to occur when the instrument is in use are followed instantaneously and without hunting. On a 50-cycle supply the current consumption is 0·6 A.

§ 5. THE NEUTRAL-DENSITY WEDGE AND ITS ADJUSTMENT

For this instrument the wedge is prepared on celluloid and is approximately 10 mm. wide and 70 mm. long. The choice of density-range must depend on the purpose of the instrument. The wedge is bent to an arc and mounted upon the moving coil of an ordinary millivoltmeter assembly*; it is held in a very light bent channel of aluminium which is carried by two supporting arms from the central clip which unites the coil, counter-weights, and pointer to the top pivot. A lower pivot in the customary clip completes the assembly, which is mounted on jewels in the usual way. The wedge is concentric with the pivot axis, but considerably above the upper pivot so that the optical path is unobstructed. The pointer is opposite to the wedge but in a lower plane. Figure 4 shows the arrangement of the coil, pointer and magnet system relative to the curved wedge.

The circuit through the coil is completed by two ligaments of thin silver-gilt strip. These exert a negligible mechanical control, and when disconnected from the instrument the movement tends to stay wherever it is placed. It is connected electrically in the place of the galvanometer in figure 2*b*. Now it will be remembered that the polarity of the valve-bridge output depends on which beam is the brighter; consequently if the wedge is in one beam it will be possible so to connect the moving coil that the wedge will be moved in the direction requisite to restore a balance. It will only cease moving when this is attained; that is, when the current through the

* This assembly without the pointer is the standard arrangement for the projected-scale instrument of the Cambridge Instrument Co., Ltd., by whom it is patented (British Patent Specification, No. 380,654).

coil is zero. The pointer-reading is then a measure of the density which has been inserted to equalize the transmissions of the two paths, and this reading will have been given by a truly null method.

Should the coil be connected in the reverse direction a balance will not be reached. On the contrary it will move away from the balance point on whatever side it happens to be and its excursion will only be limited by the stops of the instrument. This behaviour is seen if the synchronous motor is started by turning in the wrong direction, or if the phase of the grid-voltage is reversed, as was discussed in the paragraph dealing with the motor.

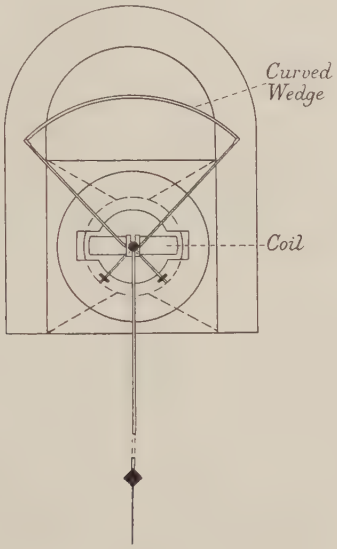

Figure 4.

§ 6. DENSITOMETER FOR LIQUIDS

An instrument incorporating the preceding features and designed to measure the optical densities of liquids will now be described. The ideal symmetrical optical arrangement of figure 1 is abandoned in favour of a more convenient lay-out, but the optical paths still form a pair of equal rectangles. The line joining K_2 and L_2 now forms the centre line of the instrument. The optical work is assembled on a stout plate which fits the case except in one corner where it is cut away to make room for the taller parts of the electrical equipment, all of which is carried on a separate sub-panel.

The scale is clear and its position makes it easy to read. Immediately below it is the milled edge of the rheostat adjustment for balancing the bridge, and the mains switch.

The instrument is used as follows. When current is first switched on the bridge will generally be found to be out of balance, but after ten minutes the valves should have reached a steady condition. Since there is no mechanical control on the

galvanometer, lack of balance of the bridge is shown by a steady drift of the pointer. The effect of an error is therefore cumulative, and adjustment can be made most precisely. This is done by moving the rheostat, which is that shown at R_3, figure 2*b*. The motor can afterwards be started and when it has come into synchronism the pointer will move to its appropriate position, coming to rest in about three seconds. To re-check the bridge-balance at any time it is only necessary to stop the motor. It is found that after several hours' working without readjustment the balance of the bridge is not sufficiently far out to cause an error of more than 0·2 of a scale division. The scale is divided into 100 divisions.

The scale-readings represent uniform changes of an approximately neutral density, and usually an experimental calibration is made to determine the relationship between the scale-readings and the concentration of a precipitate or of a coloured ion in the liquid. This is done by noting the readings when the cell, containing in succession different known concentrations of solute, is placed in the instrument.

When measurements are being made on a coloured liquid the instrument reading is a measure of the liquid-transmission for a certain band of wave-lengths, in terms of the transmission of a neutral wedge for that band, emitted by the lamp, to which the photo-cell is sensitive; consequently the relationship will change with the colour temperature of the source. The instrument under discussion is intended to operate from supply mains, and therefore changes in colour temperature are unavoidable. To circumvent this source of error a colour-filter must be used in each beam, and this filter should pass the colour complementary to that transmitted by the liquid. Care must be taken not to reduce the illumination too much in this way, or the error which might exist owing to changes of lamp-temperature will be replaced by that resulting from loss of sensitivity. The inherent colour-sensitivity of the photo-cell may in some cases suffice, but in general the use of a filter is beneficial provided that the filter is correctly chosen.

In this particular instrument illumination is provided by a 6-volt 24-watt gas-filled car head-lamp bulb, and is sufficient to give a maximum out-of-balance potential on the grids of 16 volts r.m.s., and it is found that the output from the bridge is 0·8 milliampere for an input of 1 volt r.m.s., with a load resistance 1100Ω. If T represents the transmission of the beams, T_0 can correspond to the maximum transmission. The sensitivity of the device will be represented by ΔT T. It is convenient to express T in terms of grid volts.

T, T_0

Then $$T_0 = 16 \text{ V.},$$

and $$\Delta T = 10^{-4}/0{\cdot}8 \text{ V.},$$

therefore $$\Delta T/T_0 = 8 \times 10^{-6},$$

and it follows that the sensitivity will be proportional to T. This theoretical sensitivity is never attained in practice, on account of the limitation imposed by fluctuations in the mains-voltage, but it is found that in spite of these it is possible at the maximum transmission to measure to 0·1 per cent, the possible error increasing to 1·0 per cent, at a transmission of 1 per cent, i.e. a density of 2·0.

It is worth noting perhaps that the voltage on the valve grids due to the second harmonic is 0·6 under conditions of maximum illumination.

§ 7. ACKNOWLEDGMENTS

I wish to record my gratitude to the directors of the Cambridge Instrument Co., Ltd., at whose suggestion this description has been published.

REFERENCES

The following is a list of some references to published descriptions of photometers. It does not pretend to be exhaustive.

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538.214

SOME MEASUREMENTS OF MAGNETIC SUSCEPTIBILITIES AT HIGH TEMPERATURES

By R. A. FEREDAY, Ph.D., F.Inst.P., East London College

Received October 2, 1933. *Read January* 19, 1934.

ABSTRACT. Details are given of an electromagnet which has been specially built for carrying out measurements of small susceptibilities by a method, previously described by the author, in which specially designed pole-pieces are used. It is shown that the method is applicable whether the pole-pieces are or are not saturated. An apparatus for the determination of susceptibilities at high temperatures is described in detail. It is shown that the susceptibility of anhydrous nickel sulphate can be represented by an expression of the form $(\chi - a)(T - \theta) - C$, and that anhydrous nickel cyanide, although having an exceptionally small susceptibility (of the order $3 \cdot 10^{-6}$), behaves as a normal paramagnetic substance, following a Curie-Weiss law for variation of susceptibility with temperature.

§ 1. INTRODUCTORY

In a series of earlier communications to these *Proceedings** an account has been given of the development of a new method for the determination of the susceptibilities of para- and dia-magnetic bodies at room temperatures. The method is of the non-uniform field type, in which the body under examination is suspended in a non-uniform magnetic field, the tractive force exerted upon it being measured by means of a torsion balance; and its peculiar feature is that, as a result of special design of the pole-pieces of the electromagnet producing the field, a large region exists over which the force exerted upon the body has a constant value. This overcomes the difficulty usually met with in methods based on the non-uniform-field principle, that the force exerted on the specimen varies very rapidly from point to point in the field, so that the body must be located with considerable accuracy—usually at a point of maximum force.

The only electromagnet available for the earlier work was of very unsatisfactory design, and was not capable of magnetizing the pole-pieces to saturation. A result of this was that difficulty was experienced in practice, owing to serious variations in H with variations in supply voltage; further, it was not certain that a similar field-distribution could be realized if the pole-pieces were saturated.

Through the interest of Prof. H. R. Robinson and the Metropolitan-Vickers Electrical Co., I have been able to obtain a new electromagnet designed specially for the purpose of continuing this investigation. This paper describes the instrument, and its application to an investigation of the effect of saturation of the pole-pieces on the field-distributions obtained. It was found that no essential change was

* *Proc. Phys. Soc.* **42**, 251 (1930); **43**, 383 (1931); **44**, 274 (1932).

necessary in the principle of the new method for susceptibility-measurements when the new magnet was used, and the immunity from trouble due to voltage-fluctuations enabled measurements to be made with considerably greater ease than previously.

The apparatus was modified to allow measurements to be made at temperatures up to 450° C., and an account of this new apparatus, with some results of high-temperature measurements, is included.

§ 2. THE ELECTROMAGNET

The main factors to be considered in designing the magnet were (1) ability to produce saturation of the pole-pieces when working at a gap of some 35 mm., and (2) effective cooling, since a rise in temperature would produce a change in permeability of the material of the pole-pieces, leading to changes in the magnitude and configuration of the field.

The design adopted is illustrated by figure 1. In order to allow a tapering section for the purpose of concentrating the reluctance at the gap, and at the same

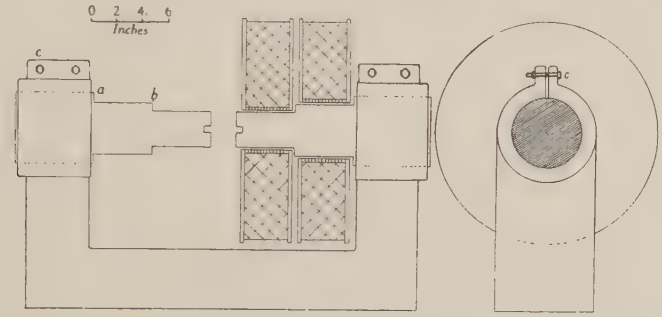


Figure 1. The electromagnet.

time to allow of free movement of the poles when the gap-width is being adjusted, the poles are stepped at *a* and *b*, and the winding on each is divided into two sections, so that the whole winding is contained on a total of four bobbins. The poles are turned to be a close sliding fit in the bored-out parts at the top of the yoke, and after adjustment of the gap-width they are locked in position by the bolts *c*. The use of this method of construction instead of the more usual screw adjustment, although it makes the initial adjustment of the gap more difficult, has the advantages of simplifying construction and maintaining better alignment of the two poles.

The yoke was cast from a pattern which was very kindly placed at my disposal by Dr J. D. Cockcroft.

The magnet is water-cooled, each of the four sections of the winding being carried by a hollow-walled bobbin which forms a complete water jacket round it. The cylindrical core of each bobbin is formed by a closely coiled spiral of copper tube, $\frac{1}{4}$ in. square, and the cheeks are built up of brass sheet and fitted with internal baffles compelling the water to flow in a tortuous path from the centre to the outside edge. Each member of the assembly—viz. each cylindrical core and each cheek—receives a separate supply of water through its own control cock from a distributor

pipe, and discharges into a common collector trough. Thus, by individual control of the different elements of the cooling system, very effective and economical cooling is possible.

The coils are wound with 0·048-in. single-cotton-covered copper wire. The turns are closely wound and are impregnated, the water-cooled bobbins being relied upon for cooling. The resistances of the coils are equal, and when running at full excitation the coils are paralleled and fed from the 240-volt mains, the power-dissipation being 2·75 A. per coil when hot, or a total of 11 A.—equivalent to 2·64 kW. Each of the smaller coils has 5700 turns, and each of the larger 6000; thus the maximum excitation is 64,400 ampere-turns.

For the work carried out so far, adequate excitation has been obtained by wiring the two outer coils in series with each other and in parallel with the inner coils; this effects economy in current-consumption, but leaves the full voltage applied to the coils nearer the gap.

§ 3. PERFORMANCE OF THE MAGNET

The magnet was found to show a very satisfactory saturation effect; figure 3 shows an (I, H) curve for one of the experimental pole-piece arrangements (see later).

Temperature rise of pole-pieces. A test was made in which a thermometer was placed in contact with the face of one of the pole-pieces, and carefully lagged to prevent loss of heat to the atmosphere. The maximum temperature-rise recorded after a long run at maximum excitation was only 3° C.

§ 4. FORCE-DISTRIBUTION

The observations of the distribution of the force $H.dH/dx$ with the new magnet formed a crucial test of the general applicability of the new method, since this represented the first test of the latter on any magnet other than that used in the original development work, and the first test with saturated pole-pieces.

The pole-piece system which had been used for susceptibility-measurements with the old magnet* was transferred to the new one, and an exploring trolley similar to that described in one of the earlier papers† was used to measure the force-distribution along the axis of symmetry of the field. Figure 2 shows the results obtained for various magnetizing currents, and figure 3 the corresponding parts of the saturation curve.

The first curve, for a current of 1·5 A. -i.e. on the steeply rising part of the curve—corresponds to the conditions under which the old magnet had always to be operated, and it will be seen that this curve is practically identical with that obtained when the same pole-piece system was used on that magnet‡. This result is of great importance as the first evidence of the reproducibility of the method.

* *Proc. Phys. Soc.* **44**, 275 (1932).
† *Proc. Phys. Soc.* **43**, 383 (1931), figure 1.
‡ *Loc. cit.* figure 1.

The next curve, for 2·5 A., corresponds to the shoulder of the saturation curve, and is of definitely changed shape; the two following curves, for 6 and 12 A. respectively, correspond to operation at saturation, and the force-distribution is changed completely, although these two curves are similar to each other.

The progressive change in the shape of the curves is very similar to the change brought about by a progressive decrease in diameter of the flat pole-piece, as may be seen by comparison with earlier curves*. This suggests that by using a slightly larger flat pole-piece, it might be possible to reproduce the flat distribution curve when working at saturation, and tests in which the same concave pole-piece was used in conjunction with larger flat ones showed that this was indeed the case.

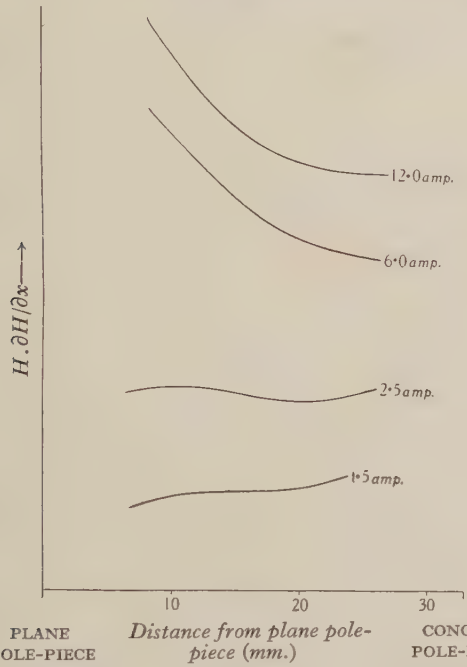


Figure 2. Force-distribution for various magnetizing currents.

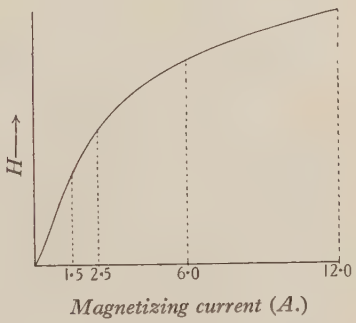


Figure 3.

Figure 4 shows some force-distribution curves obtained at saturation with the same concave pole-piece and the same gap as in the experiments just described, but with flat pole-pieces of different diameters. The distribution obtained with the 56·9-mm. flat is almost identical with that of figure 1 of the 1932 paper, the best distribution realized with unsaturated pole-pieces, and is that finally adopted for susceptibility-measurements with the new magnet.

We thus have the important result that working at saturation involves no change in the principle of the new method, or in the method to be adopted for choosing suitable pole-pieces; the only change as compared with the earlier work is that the optimum diameter of the flat pole-piece to be used with a given concave one at a given gap is slightly different.

* *Proc. Phys. Soc.* **43**, 383 (1931), figure 7.

§ 5. APPARATUS FOR SUSCEPTIBILITY-MEASUREMENTS AT HIGH TEMPERATURES

The torsion-balance T-tube used for measurements at room temperatures and illustrated in figure 2 of the 1932 paper formed the nucleus of the apparatus for high-temperature work, but whereas in the earlier work it had been found convenient to mount the tube permanently in position on the electromagnet, so that the container for the specimens was always in its correct position relative to the magnetic field, it now became desirable to be able to remove and replace the torsion balance for the purpose of changing specimens. This was mainly due to the fact that owing to the much greater diameter of the windings of the new magnet, the space between the pole-pieces was much less readily accessible than had been the case with the old design.

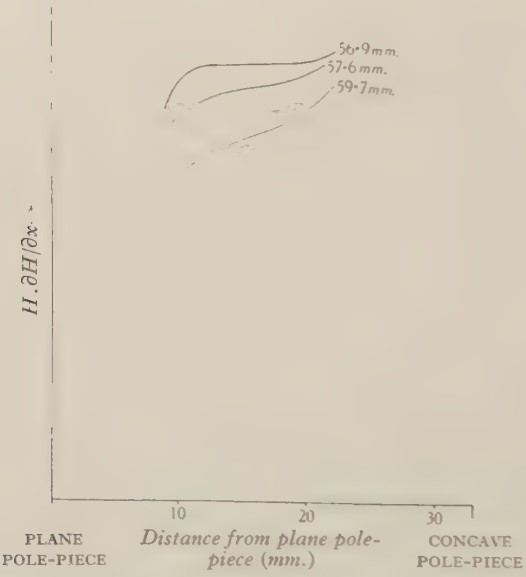


Figure 4. Force-distribution with flat pole-pieces of various diameters.

A new form of mounting had therefore to be devised, which would make it possible to remove the balance by a simple operation, and permit it to be replaced in exactly the same position relative to the field.

The arrangement adopted is illustrated by figure 5. The whole arrangement had to be designed to fit into a space 2·5-in. wide between the 18-in. diameter cheeks of the coil bobbins; it was provided with adjustments allowing the T-tube to be moved in any direction in order to make the initial adjustment to the correct position relative to the field; once this had been done, the controls could be locked, and the balance system could then be removed and replaced in exactly the same position by the operation of one easily accessible wing nut.

The main support consisted of a member, figure 5, comprising upper portions *A* and lower portions *G* working in sliding relation, to allow an adjustment for height; when this adjustment had been made the slides were locked by means of the screws *D*. This assembly was bolted, with screws passing through slotted holes in its feet, to the base of the magnet; the provision of these slots allowed an adjustment laterally—i.e. in a direction parallel to the axis of symmetry of the magnetic field. To the top of the frame was attached, by means of screws *F* passing through slots, a flat bar *E*, so that it was capable of adjustment in a direction perpendicular to the axis of the field.

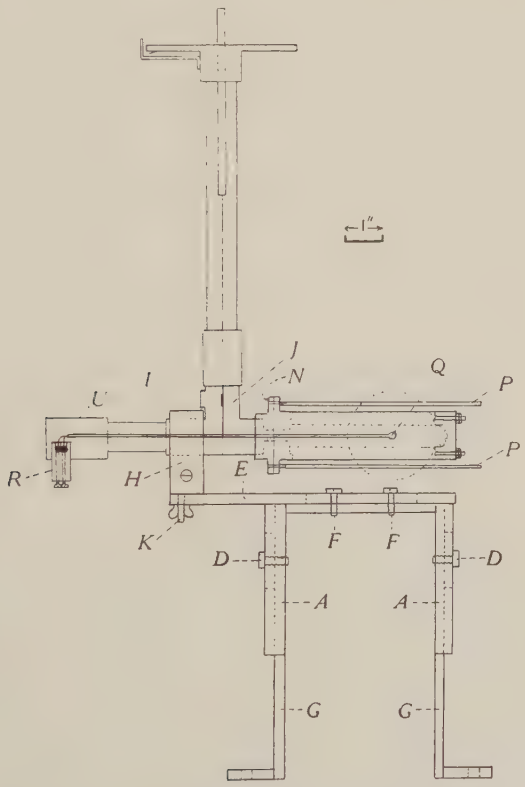


Figure 5. General view showing set-up of apparatus.

A brass block *H* was clamped to the brass part *I* of the torsion-balance T-tube *J*, as is seen in figure 5. From the base of this block projected a threaded stud *K* and two short pins, which fitted into corresponding holes at the end of the bar *E*. The engagement of these three projections with the corresponding holes ensured that the T-tube could always be attached to *E* in the same position, and it was clamped in this position by means of a wing-nut on the screwed stud.

The whole system was rigidly built up of brass bars of generous cross-section, to ensure that no distortion should be caused by the weight of the T-tube system, and it will be seen that the facilities provided for adjustment in three directions permitted the T-tube to be adjusted to a correct initial position, after which it

could readily be removed and replaced by removal of the wing-nut, which was easily accessible from the front of the magnet.

The furnace. The designing of a furnace for heating the specimens to high temperatures was a matter of some difficulty; its dimensions were strictly limited by the need for working with as small as possible an interpole gap but, at the same time, lagging had to be very effective to prevent heating of the pole-pieces; further, in the use of a horizontal furnace, there was a danger that convection currents might disturb the torsion balance and also prevent the attainment of steady, uniform temperatures in the furnace tube.

It was, however, found possible to build a horizontal furnace which behaved in a thoroughly satisfactory manner; its construction is illustrated by figure 6. The furnace tube *L* is a quartz test-tube measuring 10 cm. in length by 1·2 cm. in external diameter, on the outside of which the heater is wound. The tube is cemented firmly in position in a housing consisting of the copper tube *M* and sindanyo end-plates *N*, the space between the winding and the copper tube being filled with a lagging of

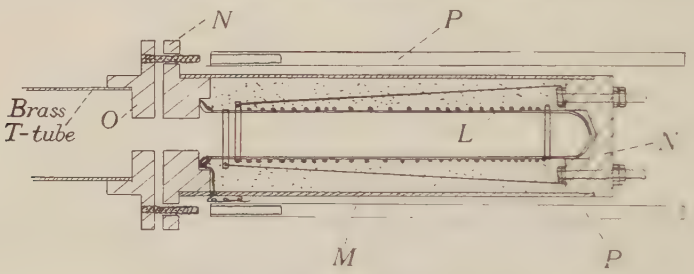


Figure 6. Section through furnace.

powdered magnesia. Sindanyo (a patent asbestos and cement board) was used because this material has very good thermal and electrical insulating properties, and can be machined accurately to any desired form. The sindanyo flange *N*, and the corresponding flange *O* which is permanently attached to the brass tube of the torsion balance, are arranged so that when the furnace is bolted to the T-tube by the bolts *P*, the only communication between the T-tube and the interior of the furnace is by way of the narrow tunnel bored through the centre of the flanges. This effectively prevents convection currents or excessive loss of heat from the furnace mouth. The arrangement of the sindanyo flange on the end of the brass T-tube also protects it from the hot part of the furnace, and prevents loss of heat to it by conduction.

The heater winding is of gauge-30 eureka wire; this material is slightly paramagnetic, having a measured susceptibility at room temperature of 8×10^{-6}. The winding comprises about fifty turns, wound bifilarly and more closely at each end of the tube than in the middle, in order to compensate for heat-losses at the ends, and to give a more uniform distribution of temperature along the tube. The winding is terminated by clamps of thin sheet copper, and as a protection is covered with a thin coating of alundum cement, which was found to have a susceptibility of $+9 \times 10^{-6}$.

The inside of the test tube is coated with a layer of copper about 0·5 mm. thick; this serves the double purpose of rendering the temperature-distribution more uniform, and of acting as an electrostatic shield between the furnace winding and the torsion-balance arm. The application of this metallic coating with a sufficiently uniform thickness presented some difficulty; the inner surface of the tube was first rendered conducting by applying a coating of aquadag (a colloidal graphite solution prepared as a lubricant for wire drawing), and after drying, baking at a low temperature, and polishing with a wad of cotton wool; this gave a very thin, highly polished coating of graphite, which served as a basis for the next process of electrolytic deposition of copper. Great care was necessary, in the initial stages of the deposition, to ensure a sufficient uniformity in the thickness of the deposit, but once a thin, uniform covering of copper had been obtained, it was easy to build up the thickness. Finally, a thin protective coating of platinum was electrolytically deposited.

Figure 5 shows how the furnace, bolted to the rear limb of the T-tube, fits between the pole-pieces of the magnet. When it is in position, there is a gap of about 1 mm. between each pole-piece and the furnace wall, and a thick sheet of mica is slipped into each of these gaps in order to protect the pole-pieces from heating by radiation or convection.

This arrangement, in combination with the water cooling of the magnet, is so effective that when the temperature in the interior of the furnace is at its maximum value of some 450° C., the temperature of the pole-pieces does not rise by more than a few degrees. An observation was made in which a small thermojunction was placed in contact with the flat pole-piece at the point where it is only separated from the furnace wall by the mica sheet; at this point the heating of the pole-pieces must be at a maximum. The measured rise in temperature when the furnace had been running at its maximum temperature for a long time was only 8° C., so that the mean temperature rise of the iron will certainly be of a negligibly small order.

The maximum temperature which can be reached with this eureka-wound furnace is, as just stated, some 450° C. This is a convenient range, since firstly the range of temperatures is also just that which can be covered with a base-metal thermocouple of fairly high voltage-output; and secondly, in the measurements which have been made of susceptibilities of nickel salts, it enables the behaviour of most simple salts to be studied up to their points of decomposition.

It is anticipated that no difficulty would be experienced in extending the measurements to considerably higher temperatures by using a furnace wound with platinum or a suitably high temperature alloy working above its Curie point, the eureka-wound furnace being retained for work at the lower temperatures. The design of the apparatus is such that furnaces are rapidly interchangeable.

Temperature-distribution. In the work of developing the furnace, a number of experimental models having differently spaced windings were wound, with the object of determining the best disposition of the turns to correct for end losses and produce a uniform temperature-distribution. For exploring the temperature-distribution in these and in the final model, as nearly as possible under working con-

ditions, a dummy torsion-balance arm was built up, carrying a thermocouple, the junction of which was contained in a small glass bulb simulating the specimen.

Figure 7 shows how, in the final form of furnace, the temperature varied along the tube for different values of heating-current. The temperature was found to be practically constant over any cross-section of the tube.

It will be seen that for every value of heating-current, a sufficiently extensive region of uniform temperature exists; at the maximum current, the temperature is constant within 1 per cent over some 15 mm. Furthermore, this region of uniform temperature always occurs at the same part of the furnace, whatever the value of the heating-current, its centre being at about 39 mm. from the closed end. Hence the apparatus had to be assembled so that the specimen whose susceptibility was to be measured always occupied that position.

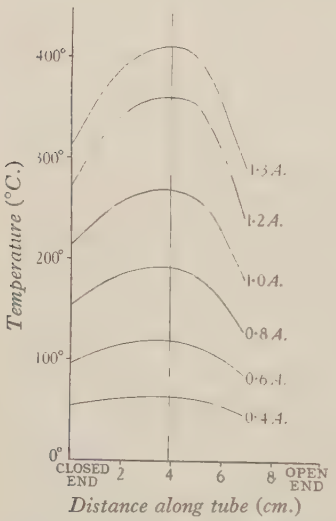


Figure 7. Temperature-distribution along furnace tube for different heating-currents.

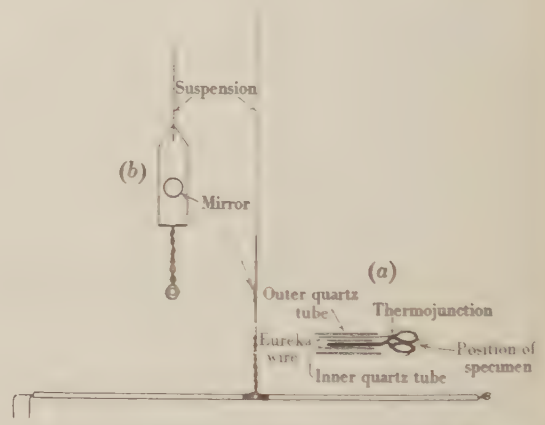


Figure 8. Suspended system of the torsion balance: (*a*) details of clamp for specimen, (*b*) front view of supporting stirrup.

The torsion balance. The torsion balance for high-temperature work was so designed that a thermocouple for measurement of the temperature of the specimen formed an integral part of it; its construction is illustrated by figure 8. The balance arm consists of a quartz tube about 1·5 mm. in diameter; through this passes a length of 30 s.w.g. silver wire terminated at one end by being bent into two small loops in parallel planes, which act as a clamp for the small bulbs containing specimens of the materials whose susceptibilities are to be measured. To one of the loops is silver-soldered a fine (36 s.w.g.) eureka wire, which is also led out inside the quartz tube of the balance arm, insulation between the two wires being provided by a finer concentric quartz tube slipped over the silver wire. This combination constitutes a thermocouple which, since it has its junction on a silver loop encircling the specimen, can be relied upon to give a reading which is a true measure of the temperature of the latter. At the other end of the arm the thermocouple wires are

bent over at right angles, to make contact with the mercury cups to be described later.

The arm is supported by a stirrup built up of copper wire and thin copper sheet, attached to its centre with shellac varnish; to the upper end of this the suspension is attached. In most of the work, a phosphor-bronze suspension strip (a galvanometer suspension) was used, but in some experiments a quartz fibre was substituted. It was attached by solder after its ends had been platinized with oil of platinum, coppered electrolytically, and tinned with molten solder, a zinc-chloride flux being used.

The exterior surface of the quartz tube of the balance arm was rendered conductive by means of a thin covering of metallic platinum, applied by coating with oil of platinum, drying, and baking. This coating is in conducting communication with the metal stirrup, and hence via the phosphor-bronze suspension with the metalwork of the T-tube, but is isolated from the wires of the thermocouple. Its purpose is to eliminate electrostatic attractions, as will be explained more fully later.

Figure 5 shows the assembly of the apparatus. It can be seen how the furnace is slipped over the specimen *Q*, and bolted to the sindanyo flange *N* on the end of the T-tube. The length of the arm of the balance is so chosen that, when the apparatus is assembled, the specimen occupies a position at 39 mm. from the closed end of the furnace tube, which, as has already been explained, is the optimum point for uniformity of temperature.

There is not much room for movement of the specimen in the narrow furnace tube, and it was therefore impossible to continue to determine the force acting on the specimen, when the field was applied, by allowing the balance to twist until equilibrium was reached, and measuring the deflection; the method of observation had to be modified by fitting a graduated torsion head and determining the twist necessary to restore the apparatus to its initial position. This adjustment to the initial position was judged accurately by reflection of a light-spot from a mirror carried on the copper stirrup supporting the balance arm.

It is arranged that deflections are generally of the order of three complete revolutions—i.e. 1080°. On the torsion head, which is 10 cm. in diameter and graduated in degrees, a fraction of a degree can easily be estimated, and no difficulty is experienced in reading the deflection to within 0·1 per cent. The actual observation could, in fact, be made with greater accuracy, but this would not represent such accuracy in the susceptibility-measurement, because of slight elastic after-effects in the suspension, and imperfections in the set-up of the apparatus.

At the forward end of the T-tube can be seen the arrangement for making contact between the thermocouple wires and the external bridge circuit; the details will be appreciated by reference to figure 9.

The essential part of the apparatus is an ebonite block *R* from which part is cut away as shown. Three saw-cuts *S* were made, and between each pair of cuts a hole was drilled through to the base of the block and tapped to take a 4 B.A. screw *T*. A sheet of mica was slipped into each cut and cemented in position with shellac varnish; in the cut-out part of the block the space between these mica sheets forms

two small cells which are used as mercury cups, electrically insulated from each other but in good thermal communication, since they are separated only by a thin sheet of mica. With screws T inserted in the tapped holes, and screwed in fully, the cups are filled with mercury, the dimensions of the cups and the threaded holes being such that when the screws are unscrewed nearly to their fullest extent, the mercury runs down into the threaded holes in the ebonite leaving the cups empty.

The position of the block at the end of the T-tube is such that the turned-down ends of the thermocouple wires come into the mercury cups. When it is desired to make a measurement of temperature, the screws are screwed up to fill the cups with mercury, making contact between these wires and silver and eureka leads connecting the cups with the cold junction and the bridge circuit. When an observation of magnetic force is being made, the cups are emptied by withdrawing the screws to the necessary extent, and the ends of the thermocouple wires then swing freely in the empty cups.

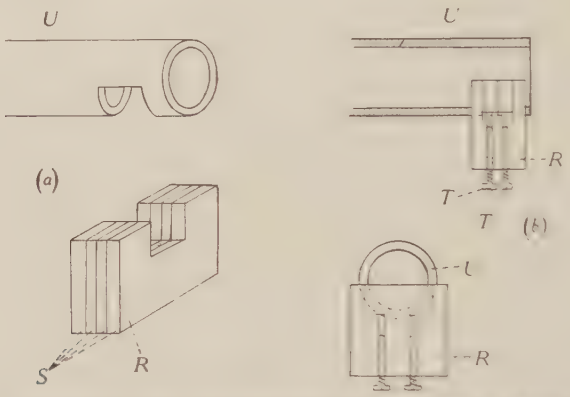


Figure 9. Thermocouple contact block. (*a*) Perspective view of brass tube and ebonite block, before completion. (*b*) Section and front elevation of completed apparatus.

The ebonite block R is supported in a massive tube U of brass, and this, combined with the fact that the cups are in good thermal contact through the thin sheet of mica separating them, ensures that no appreciable difference exists between the temperatures of the cups.

The cold junction is kept in ice. After having been set up, the thermocouple was very carefully calibrated with reference to standard fixed points; over the range 0–100° C calibration was effected by comparison with a standard mercury thermometer. After the torsion balance had been set up, direct fixed-point determinations could not conveniently be made. The calibration was therefore checked from time to time by comparison with another calibrated thermocouple, and was found to be maintained in a perfectly satisfactory way; measurements of the output voltage of the couple were made on a specially constructed slide wire bridge.

In order to guard against the introduction of parasitic electrostatic forces acting on the torsion balance, the whole of the metalwork—torsion head, thermocouple contact tube, T-tube, metallized torsion balance arm, and internal shield of the furnace tube—was bonded together and earthed.

§ 6. METHOD OF OBSERVATION

It should be clear from the foregoing descriptions that one of the main features of this apparatus and method is that a tiresome adjustment of each specimen is replaced by one initial adjustment of the apparatus, so that when once this has been effected, the different specimens can be slipped into position on the torsion balance and observations can immediately be commenced. This initial adjustment of the apparatus is obviously of fundamental importance, and must be carried out with care.

Two main adjustments have to be made when the apparatus is set up. (i) The torsion balance must be so mounted that the specimen always occupies a position on the axis of symmetry of the field, and in the region of uniform force. (ii) The furnace must be so related to the torsion balance that the specimen always occupies a position in the region of maximum uniformity of temperature.

As has been explained in the preceding section, the latter condition is satisfied when the apparatus is being built, by a suitable choice of the length of the arm of the torsion balance. In making the initial adjustment to satisfy the first condition, a specimen was placed in position on the torsion balance, which was mounted in position between the pole-pieces of the magnet. By adjusting a wire rider on the forward part of the balance arm, the arm was set in a horizontal position; and by means of the various adjustments provided on the torsion-balance support, the specimen was brought to a position at the centre of the axis of symmetry of the field, the torsion head being set so that the balance arm settled along the axis of the horizontal arm of the T-tube. The lamp and horizontal scale system used for setting the zero of the torsion balance was now set so that the spot indicated zero on the scale, and all the controls were locked. If now the torsion balance was removed and replaced, or the specimen changed, any displacement from the axis of symmetry in a vertical direction was indicated by the spot shifting to a position above or below the scale.

With the furnace in position over the specimen, the latter's movements were restricted to a matter of some 3 mm. on each side of the mid-point of the gap, which, as can be seen from an inspection of figure 4, meant that it was confined to the region of uniform force. Thus it was only necessary, in setting a specimen, to slip it on to the end of the torsion balance so that it was clamped in position between the two silver loops, bolt on the furnace, place the whole apparatus in position on the electromagnet, and adjust the rider to bring the light spot on to the scale.

Owing to slight differences in the sizes of the specimens, the distance between the centres of the specimen and the torsion balance arm—i.e. the true arm of the couple measured—was liable to slight variation. These variations were, however, never more than 0·5 mm., the length of the arm being 113 mm. A correction was applied, and was of the order 0·5 per cent.

As containers for the specimens, small quartz bulbs some 3 mm. in diameter were used. In some cases it was necessary to seal off the necks of the bulbs, and glass was then substituted, as it was not found possible to seal a quartz bulb without

damaging the contents by heat. A blank experiment was in every case performed on the empty bulb, in order that a correction might be applied for the magnetism of the bulb and balance arm over the temperature-range concerned. This correction amounted in general to some 5 per cent. After this experiment had been carried out the bulb was filled with the powdered salt, weighed, sealed if necessary, and replaced on the torsion balance; measurements of susceptibility at various temperatures were then proceeded with.

When susceptibility-measurements were being made the procedure was as follows: the specimen having been inserted and the apparatus placed in position on the magnet, the rider was adjusted to bring the light spot on to the scale, which indicated that the specimen was on the axis of symmetry of the field. After verifying that the balance arm was swinging freely, the readings of the torsion head and light spot were noted. The field was then applied, and the torsion head turned to restore the light spot to its initial position; thus the twist necessary to compensate for the magnetic couple was determined. The field was then switched off and the torsion head restored to the zero position; the heating-current was switched on at some predetermined value and the mercury cups were filled to connect the thermocouple to the bridge circuit. Thus the temperature of the specimen could be measured, and when it had reached a steady state the mercury was withdrawn from the cups so that the balance arm could swing freely, and another twist-measurement was made as before. The heating-current was then set to a higher value, and the observations were repeated. Proceeding in this way, a series of twist-measurements was made over the range of temperature desired.

The measurements were corrected for the twist necessary to compensate the forces acting on the bulb and the balance arm by applying the results of the preceding experiment on the empty bulb.

θ Now we have, if the corrected twist be θ,

$$\text{Couple acting on the specimen} = n\theta = rm\chi H.\partial H/\partial x,$$

n where n is the restoring couple of suspension per unit twist;

r r the arm of couple;

m m the mass of specimen; and

χ χ the susceptibility of specimen.

Thus
$$\chi = \frac{n}{rH\partial H/\partial x} \cdot \frac{\theta}{m} = K.\frac{\theta}{m},$$

K where K is a constant characteristic of the apparatus. All results are therefore expressed in terms of $\theta\, m$, conveniently expressed in "degrees of twist per milligram." These values are proportional to the susceptibilities, and can be converted to susceptibility-values when K is known. K can be determined from the known constants of the apparatus, but this is a somewhat involved process when accurate results are required, and it is preferable for most purposes to effect a calibration of the apparatus by the use of some substance whose susceptibility is already known with accuracy.

§ 7. MEASUREMENTS OF SUSCEPTIBILITIES OF SOME NICKEL COMPOUNDS

Anhydrous nickel sulphate. This compound has been examined over the range 14–290° K., by L. C. Jackson*, who found that over the upper part of this range it followed a Curie-Weiss law with θ equal to $79{\cdot}4^{\circ}$, and p to 16·9 Weiss magnetons. Honda and Ishiwara† made observations up to the temperature of decomposition (about 800° C.) and found that the (χ^{-1}, T) curve showed a slight downward curvature.

More recently, Cabrera and Duperier, and Mlle Serres‡ have suggested that the less simple law

$$(\chi - a)(T - \theta) = C$$

will explain the behaviour of the salt at high temperatures, and Serres assigns to a the value $425 . 10^{-6}$, without, however, giving more detailed numerical results.

In the present work, observations have been made over the range 290–550° K. The material used was prepared from a pure heptahydrated sulphate, which, after recrystallization a few times from conductivity water, was dehydrated by being heated in a quartz vessel in an electrically heated quartz tube furnace. After the quartz bulb in which the specimen was introduced into the magnetic field had been filled, it was again heated for a time in order to remove any water acquired during the filling operation. Two samples of the hydrated sulphate were obtained, these being from entirely different batches received from the manufacturer.

The results of the observations were very consistent; curve A of figure 10 shows the results for a specimen from each batch, and it will be seen that they are in excellent agreement.

In the absence of an accurate calibration of the apparatus, absolute values have been assigned to the susceptibilities by adjusting the observed relative values to agree with Jackson's value at 290° K. This leaves a little uncertainty as to the exact value of χ, but it allows a test to be made of the manner in which the expression of Mlle Serres expresses the nature of the variation of χ with T.

The results of Honda and Ishiwara over the temperature range concerned are also shown on the curves; these are seen to be in very good agreement, giving good justification for the calibration adopted.

It is obvious that the curve showing the relation between χ^{-1} and T shows a concavity towards the T axis, indicating a departure from a true Curie-Weiss law. In order to determine whether a relation of the form suggested by Serres, viz.

$$(\chi_m - 425 . 10^{-6})(T - \theta) = C,$$

will explain this departure, a second curve B, figure 10, has been prepared, showing the relation between $1/(\chi_m - 425 . 10^{-6})$ and T; it will be seen that this gives a good linear relation, and it may be concluded that the observations are in good agreement with the law suggested by Mlle Serres.

χ_m

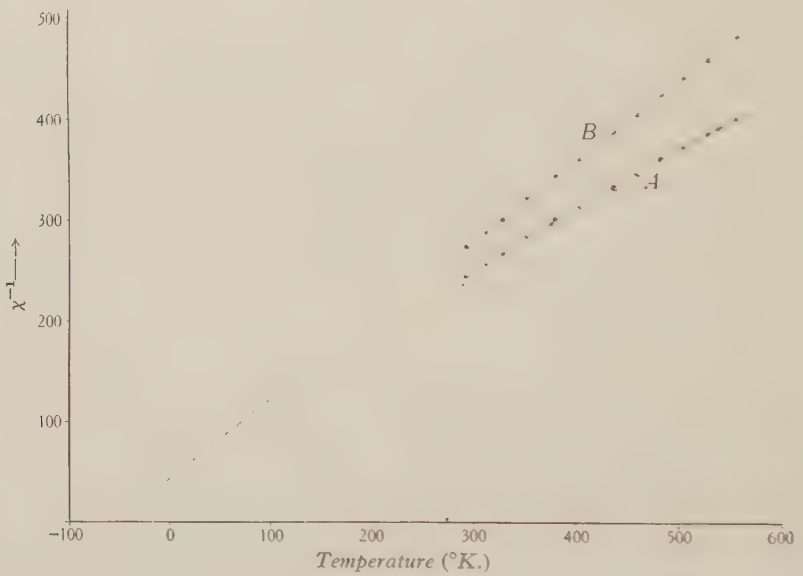
* *Phil. Trans.* A **224**, 1 (1923).
† *Sci. Rep. Univ. Tohoku*, **4**, 215 (1915).
‡ *Bull. Soc. Franç. de Phys.* **4**, 74 (1933).

The value of θ indicated by the curve is $-58°$; and on the basis of the calibration in terms of Jackson's figures, the value deduced for the magnetic moment p of the carrier from the value of C in the above expression is 16·0 Weiss magnetons.

Table 1 has been reproduced to show the manner of reducing the results; it refers to specimen *a* of the curves.

Table 1

T (°K.)	A (°)	B (°)	C (°)	D	χ_m	χ_m^{-1}	$\chi_m - 42.10^{-6}$	$(\chi_m - 42.10^{-6})^{-1}$
294·0	823	815	818	40·9	0·00412	243	0·00379	270
312·7	779	776	779	39·0	0·00393	254	0·00351	285
351·6	701	705	708	35·4	0·00356	281	0·00314	318
403·0	629	641	643	32·2	0·00324	309	0·00282	355
458·0	563	582	584	29·2	0·00294	340	0·00252	397
502·0	516	539	541	27·1	0·00273	366	0·00231	433
554·4	474	501	503	25·2	0·00254	394	0·00212	472



× specimen *a*; ⊙ specimen *b*; ▲ results of Honda and Ishiwara.

Figure 10.

Column A gives the observed twist, as explained on p. 226. B gives this value corrected for the twist needed to compensate for the magnetism of the bulb and support. In column C the further small correction for variation in the arm of the couple, explained on p. 225, is applied. In column D this value is reduced to a relative susceptibility in terms of twist per milligram by dividing by the weight of the specimen; and this is brought to a true susceptibility, on the basis of Jackson's figures, in the next column.

Nickel cyanide. The properties of this salt are particularly interesting, and it had already been selected for a special examination. It is known that the susceptibility

of the hydrated salt is of the same order of magnitude as those of other simple nickel salts, but that on dehydration its value decreases very considerably*. Bose has suggested that "on complete dehydration the compound would be found to be diamagnetic." On the other hand, Mlle Serres† finds that the anhydrous cyanide is paramagnetic and "seems to obey a Weiss law without correction for a constant paramagnetism."

For this work, samples of the cyanide were prepared by precipitation, carefully prepared solutions of nickel acetate and potassium cyanide of AR quality being used. The precipitate was thoroughly washed, and dried at 100° C. It was then found to contain 23·11 per cent of water, which corresponds to a formula $Ni(CN)_2 + 1{\cdot}85\ H_2O$ and points to a partly dried dihydrate.

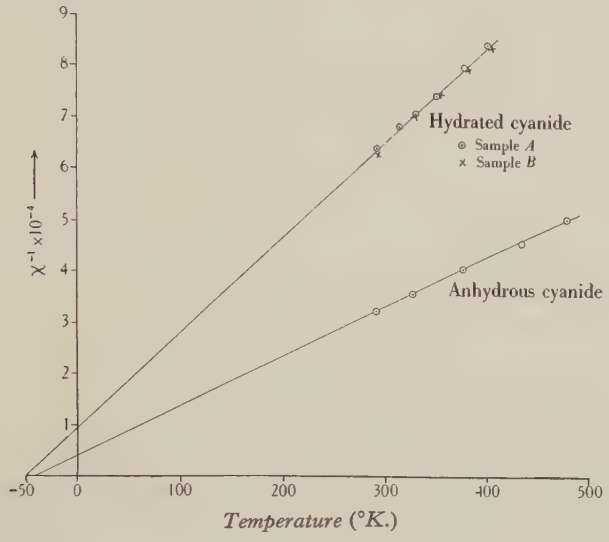


Figure 11.

Measurements on hydrated salt. Measurements were made on several samples of this compound, with consistent results; observations were made on samples in both open and sealed bulbs, the latter in case any dehydration might be occurring at the higher temperatures, but it was found that similar results were obtained in each case. In the curves of figure 11, results are given for a sample contained in a closed bulb (A) and an open one (B); the χ^{-1} results are a trifle higher for A than for B, but the general agreement is entirely satisfactory. The reduction to absolute values is here again based on Jackson's figures for the anhydrous sulphate. A Curie-Weiss law is followed with T equal to $-50°$, and the magnetic moment of the carrier, evaluated on the basis of the composition $Ni(CN)_2 + 1{\cdot}85\ H_2O$, is 12·3.

Measurements on anhydrous salt. In examining this compound, two methods were available. (i) An open-necked bulb was filled with the hydrated cyanide, and placed

* Bose, *Nature*, **125**, 708 (1930); Fereday, *Proc. Phys. Soc.* **44**, 279 (1932).
† *Loc. cit.*

in position in the torsion balance; observations of susceptibility were made at temperatures up to just above the point of dehydration (about 200° C.). The temperature being kept constant (susceptibility, time) observations were then made, and these enabled the dehydration process to be followed until the attainment of a constant value indicated complete dehydration. Measurements of the susceptibility of the dehydrated salt were then made at successively lower temperatures down to room temperature. (ii) A glass bulb was filled with the hydrated cyanide, which was then dehydrated in the torsion balance as before. By effecting dehydration in this way, a very close control could be exercised. When dehydration was complete the bulb was removed and its neck was immediately sealed by the application of a fine, pointed blowpipe flame. It was then replaced in the torsion balance for the purpose of making the susceptibility-measurements.

A number of observations were made by each method, and good agreement was obtained. The second method proved the more satisfactory, however, since with an open-necked bulb there was a tendency for rehydration to occur at the lower temperatures, resulting in a slight increase in susceptibility.

A specimen (χ^{-1}, T) curve is reproduced in figure 11. The sample to which this refers was prepared by dehydration and sealing of the hot bulb. It is seen quite definitely that the anhydrous salt behaves as a normal paramagnetic, following a Curie-Weiss law; and definite values can be assigned to the constants θ and p. θ is seen to have the value -42°; and p, calculated on the basis of the formula $Ni(CN)_2$, is 4·75 Weiss magnetons.

§ 8. ACKNOWLEDGMENTS

The very satisfactory performance of the electromagnet used for this research is due almost entirely to the interest of the Research Staff of the Metropolitan-Vickers Electrical Co., who designed and built the magnet, and I should like to take this opportunity of expressing my appreciation of their assistance.

I am indebted to Prof. H. R. Robinson for the facilities which he has placed at my disposal in the Physical Laboratories of the East London College; and to the Department of Scientific and Industrial Research, who have supported the work by a Senior Research Award.

535.822.01

THE THEORY OF THE MICROSCOPE, II: DARK-GROUND ILLUMINATION

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ABSTRACT. This paper applies the methods developed in a previous paper* to the discussion of the effects in dark-ground illumination when the image of the source of light is projected into the object plane by an illuminator of the symmetrical type. The treatment is two-dimensional. The conditions necessary for the formation of genuine and spurious images are investigated, and it is shown that the Abbe principle is theoretically valid in the cases considered. A short practical investigation with Grayson's rulings supports the theoretical conclusions, but indicates the desirability of closer examination of the causes of misleading interference phenomena.

§ 1. SCOPE OF THE PRESENT INVESTIGATION

A previous paper on the theory of the microscope effected the analysis of the mode of the image-formation for non-self-luminous objects in a manner distinct from that used by Abbe. If the object is illuminated by light focused by a condenser so that the image of the source falls in the image plane, the distribution of illumination due to any elementary point of the source becomes known, and it is possible to calculate the distribution of illumination in the posterior focal surface of the objective when the object consists of a series of equidistant apertures. Further steps in that paper effected the calculation of the interference phenomena in the final image plane; firstly those due to a single point of the source, and finally the complete effect due to an extended source of light. The treatment throughout was two-dimensional.

In recent years great importance has been attached to the dark-ground method, in which the object is illuminated from the condenser by oblique rays with numerical apertures greater than that of the objective. The light can only reach the image by the effects of reflection, refraction, or scattering, in which the path of the energy may be partly deflected into the objective. The phase relations of the intercepted vibrations have an essential bearing on the actual distribution of light into the optical system.

The scope of the present investigation is to extend the methods described above to the case of dark-ground illumination, and to discuss how far the results may be applied to practical cases. The problem has been discussed by M. J. Cross† on the basis of the Abbe principle, but his assumption of an oblique illumination by a

* *Proc. Phys. Soc.* **43**, 186 (1931).
† *Knowledge*, 1912, p. 37.

narrow parallel beam of light seems artificial at first sight, since most modern dark-ground illuminators give an illuminating beam of a conical character, or rather a beam contained between two cones, and yield an image of the source of light in the image plane. Although the three-dimensional case presents undue difficulty, it is possible to give a discussion of the two-dimensional case which resolves some of the difficulties.

Consider one point only of the source of light B, figure 1. The condenser, which will be first supposed to give an unrestricted beam, focuses this in the object plane. A_2 Let the illuminating beam have a maximum numerical aperture A_2, let the radius f of the wave surface (of circular section) as it leaves the condenser be f, and let the a amplitude be a. The discussion must be limited to the case in which the maximum numerical aperture is small.

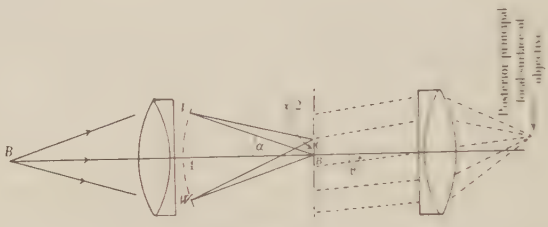

Figure 1.

Consider the amplitude, in the neighbourhood of the image point B', contributed α by an element of the wave surface at I' from which the ray $I'B'$ makes an angle α with the principal ray AB'. The width of the element of the wave surface subtending δs an angle $\delta\alpha$ is $f\delta\alpha$ and the gross amplitude contribution δs at B' is therefore given by

$$\delta s = \frac{a}{\lambda f} \cdot f\delta\alpha.$$

There will be a contribution of equal numerical magnitude from the corresponding point symmetrically situated in the other half of the wave. If we consider vibrations occurring in the plane of the diagram, it is clear that the resultant vibrations at B' will be polarized in directions inclined to the image plane at angles $-\alpha$ and $-\alpha$ respectively. The amplitude contributions of these components will therefore be subject to a correcting factor of $\cos\alpha$, since the resultant vibration must be in the image plane. Hence the net amplitude contribution δs for each of these vibrations is given by

$$\delta s = \frac{a}{\lambda f} \cdot f\delta\alpha \cos\alpha = \frac{a}{\lambda_0}\,\delta(n \sin\alpha),$$

n, λ_0 where n is the refractive index of the space concerned, and λ_0 is the wave-length of light in a medium whose refractive index is unity. We may write the above A expression, since $n \sin\alpha$ is A the numerical aperture,

$$\delta s = \frac{a}{\lambda_0}\delta A.$$

When we consider the vibration components perpendicular to the plane of the

diagram, we may recall that the amplitudes of all radiations reaching the image through the outer regions of any optical system are subject to uncertain losses through the more oblique reflections encountered by the marginal rays, and it would be very difficult to take account of such effects. We can be sure however that the $\cos\alpha$ reduction, taken for the vibrations as a whole, is quite in harmony with a possible case; it will approximate to unity under the condition that the angles involved are small. This will now be assumed.

Vibrations from V and W arrive at B' in the same phase, but at C, at a distance $\frac{1}{2}x$ from B', the disturbance from V has a shorter path, while that from W has a longer one. The relative phase angles (x being very small) are x

$$\pm \lambda_0^{-1} \pi n x \sin\alpha.$$

The amplitude s_0 due to the whole symmetrical wave of numerical aperture A_2 is given by

$$s_0 = \frac{2a}{\lambda_0} \int_0^{A_2} \cos\left(\frac{\pi x}{\lambda_0} A\right) dA,$$

but if the central regions of the wave are screened to produce symmetrical dark-ground illumination between numerical apertures A_1 and A_2, the amplitude at C is now

$$s = \frac{2a}{\lambda_0} \int_{A_1}^{A_2} \cos\left(\frac{\pi x}{\lambda_0} A\right) dA,$$

$$= 2a(\pi x)^{-1} \{\sin(\pi x A_2/\lambda_0) - \sin(\pi x A_1/\lambda_0)\}.$$

Write $p = \pi x/\lambda_0$; then p

$$s = 2a(\lambda_0 p)^{-1} \{\sin(pA_2) - \sin(pA_1)\}.$$

In the same manner as that of the foregoing paper, we may now consider an object consisting of a series of equidistant apertures at relative distances x, and placed at distances $\pm x/2$, $\pm 3x/2$, $\pm 5x/2$, etc., from the centre point of illumination. These apertures will become the sources of disturbances having the same phases, but varying amplitudes given by the corresponding value of s in the above expression.

These vibrations are focused by the objective of the microscope, and we may consider the effects at points, in the principal focal surface behind the objective, corresponding to various directions of divergence from the normal through the point B'. For a small angle of diffraction, θ, from the normal to the object plane, θ
the resultant amplitude for the first pair of apertures on either side of B' will be proportional to

$$4a(\lambda_0 p)^{-1} (\sin pA_2 - \sin pA_1) \cos\{(\pi x n' \sin\theta)/\lambda_0\},$$

where n' is the refractive index on the side of the object plane towards the objective.

The resultant from all the successive pairs of apertures is therefore given (when A_θ is written for $n' \sin\theta$) by: A_θ

$$\text{Amplitude}_\theta = \text{constant} \times 4a(\lambda_0 p)^{-1}\{(\sin pA_2 - \sin pA_1)\cos pA_\theta$$
$$- \tfrac{1}{3}(\sin 3pA_2 - \sin 3pA_1)\cos 3pA_\theta$$
$$+ \tfrac{1}{5}(\sin 5pA_2 - \sin 5pA_1)\cos 5pA_\theta + \text{etc.}\},$$

$$= \text{constant} \times 4a(\lambda_0 p)^{-1}\{(\sin pA_2 \cos pA_\theta + \tfrac{1}{3}\sin 3pA_2 \cos 3pA_\theta$$
$$+ \tfrac{1}{5}\sin 5pA_2 \cos 5pA_\theta + \text{etc.})$$
$$- (\sin pA_1 \cos pA_\theta + \tfrac{1}{3}\sin 3pA_1 \cos 3pA_\theta + \tfrac{1}{5}\sin 5pA_1 \cos 5pA_\theta + \text{etc.})\},$$

$$= \text{constant} \times 2a(\lambda_0 p)^{-1}[\{\sin p(A_2 + A_\theta) + \tfrac{1}{3}\sin 3p(A_2 + A_\theta) + \text{etc.}\}$$
$$+ \{\sin p(A_2 - A_\theta) + \tfrac{1}{3}\sin 3p(A_2 - A_\theta) + \text{etc.}\}$$
$$- \{\sin p(A_1 + A_\theta) + \tfrac{1}{3}\sin 3p(A_1 + A_\theta) + \text{etc.}\}$$
$$- \{\sin p(A_1 - A_\theta) + \tfrac{1}{3}\sin 3p(A_1 - A_\theta) + \text{etc.}\}].$$

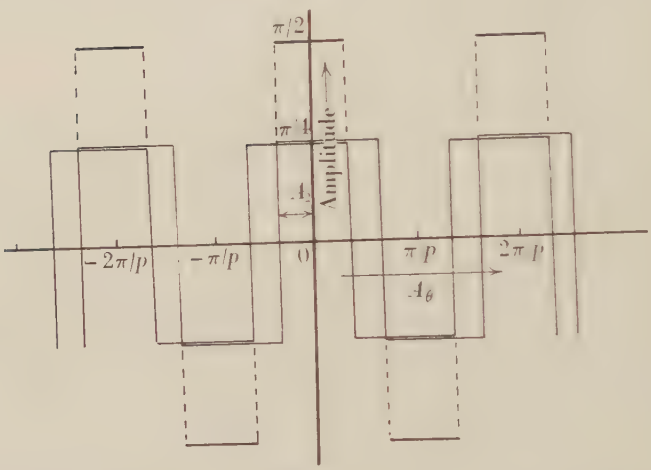

Figure 2.

In actual microscopical observations, objects of fine regular structure are often encountered. A diatom may have some hundreds of regularly spaced ribs or dots. A diffraction grating having 30,000 lines to the inch could easily be resolved by an objective of numerical aperture 0·65, and it is quite possible for, say, 1000 elements of the grating to contribute their effects towards the formation of the diffraction maxima in such a case. Practical considerations therefore suggest that cases of theoretical interest include those in which a very great number of terms in the above series are taken; and expressing the result of the series as its sum taken to infinity then involves only a slight error.

The first Fourier series in the brackets has, if an infinite number of terms be taken, corresponding to a great number of apertures, a value of $\pi/4$ from $p(A_2 - A_\theta) = 0$ to $p(A_2 + A_\theta) = \pi$, and a value of $-\pi/4$ when the same angle lies between π and 2π, and so on. The other brackets can be similarly interpreted. The first bracket is thus $\pi/4$ from $pA_\theta = pA_2$ to $pA_\theta = \pi - pA_2$, and the second bracket is $\pi/4$ from $pA_\theta = pA_2$ to $pA_\theta = -(\pi - pA_2)$.

The first two series alone represent the case when the full aperture of the condenser illuminates the object plane. Their sum represents a series of isolated maxima (see figure 2) extending over regions corresponding to numerical apertures $\{-(\pi/p) \pm A_2\}$, $\{\pm A_2\}$, $\{(\pi/p) \pm A_2\}$, etc.; they constitute the diffraction maxima produced by the object. The range of numerical aperture of each maximum is equal to that of the condenser. The spacing of the maxima depends only on π/p, i.e. on λ_0/x. It is clearly seen that

$$n' \sin\theta = \lambda_0/x$$

is the usual condition giving the angle of diffraction of the first-order spectrum for normal illumination.

If the last two brackets in the full expression above are evaluated, they are found to give similar isolated maxima extending over the numerical apertures

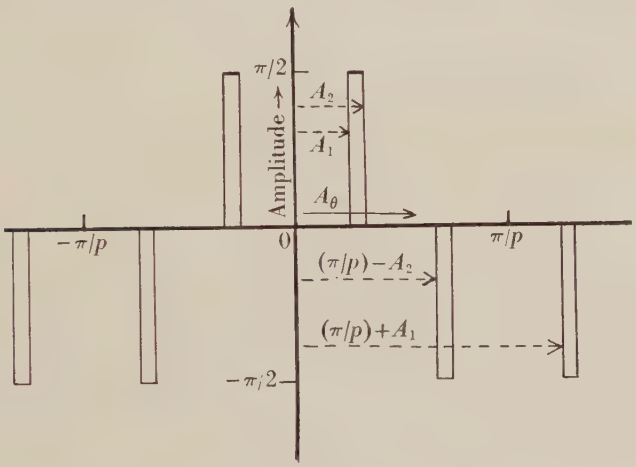

Figure 3.

$\{-\pi/p \pm A_1\}$, $\{\pm A_1\}$, $\{\pi/p \pm A_1\}$, and they have the opposite signs to the first two brackets. The final resultant therefore shows a succession of maxima symmetrically placed about the apertures $-\pi/p$, 0, π/p, etc., the centres of the maxima being found at $\pm \frac{1}{2}(A_2 + A_1)$ on each side, and the width being $A_2 - A_1$ in each case. The maxima symmetrical about the centre correspond to the direct light from the condenser.

In the case of symmetrical dark-ground illumination the lower aperture A_1 of the illumination is greater than the aperture, A_ϕ say, of the objective, and the direct light cannot enter. We have, however, only to increase A_2 until

$$(\pi/p) - A_2 < A_\phi$$

before light from the first-order diffraction maxima on each side begins to enter the objective. This condition can be written

$$\pi/p < A_\phi + A_2$$

or

$$\lambda_0 < x(A_\phi + A_2).$$

If A_2 is only slightly greater than A_ϕ, then we shall have the approximate limiting condition that

$$\lambda_0 = 2xA_\phi$$

or

$$x = \text{grating spacing} = \lambda_0/2A_\phi,$$

and in this case symmetrical parts of the two first-order maxima of opposite sign will be just within the limit of the numerical aperture of the objective, one on each side. The condition is suggested diagrammatically in figure 4. In the ideal case which we have discussed, the light in all the maxima will be coherent, and there would (in the case of a single-point source of illumination) be reason to expect an interference effect which would be seen in the microscope as a succession of fringes of a frequency corresponding in this case very nearly to the resolution of which the objective is capable with direct oblique illumination. If, on the other hand, the spacing of the apertures is imagined to vary while the disposition of the illumination remains constant, the mean numerical aperture of the diffracted beams entering the objective will be $(\pi/p - A_D)$, where $A_D = \frac{1}{2}(A_1 \quad A_2)$. Now A_D is not very different from A_ϕ in the case being considered, so that

$$\pi/p - A_D = \pi/p - A_\phi \text{ approximately,}$$

and these beams will both take the axial direction when

$$\pi/p = A_\phi, \text{ i.e. when } A_\phi = \lambda_0/x.$$

There is evidently no correspondence between any possible interference effects from these first-order maxima and the spacing of the object elements, since the interference fringes will grow indefinitely broader as these maxima both approach the axis and coincide. It will thus be expected that even if any of the maxima can produce mutual interference effects, the fringes will not correspond in any way to the geometrical image.

On the other hand the obliquity of the cone of illumination from the condenser may be just great enough to allow corresponding first-order and second-order maxima on the same side both to enter the objective. If they enter symmetrically at apertures $\pm A_\phi$, the entry of the first-order maximum is represented by

A_ϕ

$$-(\pi/p) + A_D = A_\phi \qquad \ldots\ldots(1),$$

while the entry of the second-order maximum is represented by

$$-(2\pi/p) + A_D = -A_\phi \qquad \ldots\ldots(2).$$

The condition is suggested in figure 5, but the beams are only shown as unilateral for simplicity. On subtracting the equations we find

$$A_\phi = \pi/2p = \lambda_0/2x,$$

i.e.

$$x = \lambda_0/2A_\phi,$$

so that once more a lens of numerical aperture A_ϕ could attain its full resolving power; but the same equations indicate that in order to attain this condition $A_D = 3A_\phi$, i.e. three times the numerical aperture of the objective. This case was worked out by M. J. Cross, but it has been judged worth while to re-state the

argument in the present connection. The separation of the interfering maxima is π/p, which only depends on the spacing of the apertures. The interference phenomena in the image correspond in frequency to the object (see below, p. 240).

A practical test such as that described below soon shows that the right-hand maxima will not give observable interference effects with the left-hand maxima under the ordinary conditions in the microscope, though effects of the kind could probably be observed with special apparatus. The reason for the non-appearance of the effects is explained by the theory given below.

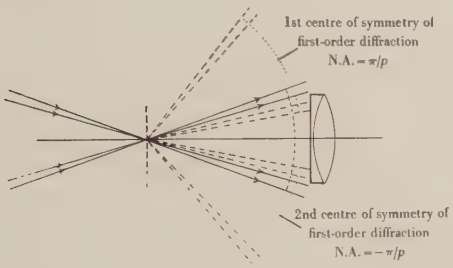

Figure 4.

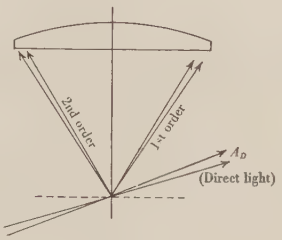

Figure 5.

§ 2. GENERAL THEORY OF THE DISTRIBUTION OF LIGHT IN THE IMAGE OF A SERIES OF APERTURES (BRIGHT-GROUND CASE)

With reference now to the first paper, an expression was given on page 198 for the phases and amplitudes of the maxima appearing in the back focal surface of the objective in the case when the object consists of a series of indefinitely narrow apertures lying at points defined by

$$u,\ u \pm v,\ u \pm 2v,\ \text{etc.},$$

where v is a measure of the grating interval and u is a corresponding measure of the displacement of the maximum illumination, in the grating plane, from the central aperture of the grating. The symbol u actually represents a phase angle, being $2\pi h\ (NA_c)/\lambda_0$, where h is the linear interval corresponding to u, and NA_c is the numerical aperture of the cone of illumination from the condenser. The symbol v represents the corresponding phase angle expressing the measure of the grating interval x, i.e. it is $2\pi x\ (NA_c)/\lambda_0$. The numerical aperture NA_c will be written A_c for brevity in subsequent expressions.

v, u

h

A_c

The expression for the displacement is

$$(\pi/v)\ (1^* + \Sigma e^{is_r u^*}),$$

where $s_r = 2\pi r/v$, and the meaning of the *asterisk* is that each term is subject to the limitation that it only exists within the limits $m + s_r = \pm 1$; m is the measure of the obliquity of the diffracted light. It should be remembered that the expression should be multiplied by an infinitesimal quantity du, where du represents the width of a (very narrow) aperture, but this need not be written in the subsequent expressions unless it is desired to integrate for apertures of finite width.

$*$

s_r

m

Starting from this expression we may notice a method of deriving the amplitude in the final image which is more systematic than that given on page 203 of the original paper.

The result quoted above gives the amplitudes and phases of the maxima in a reference surface centred in that point B'' in the final image plane which is conjugate to the point of maximum illumination in the object plane; suppose that the

$\phi(P)$ complex $\phi(P)$ represents the amplitude and phase of the disturbance at a point in this spherical reference surface, corresponding to a numerical aperture A' equal to $n' \sin \alpha'$ on the image side. Now in the case of a uniform circular wave-front centred in B'', figure 6, we can write the relative phase angle of the disturbance (from the region of the wave-front defined by the angle α') arriving at C'' in the

h' image plane at a distance h' from B'' as

$$\lambda_0^{-1} 2\pi n' h' \sin \alpha' \quad \text{or} \quad \lambda_0^{-1} 2\pi h' A',$$

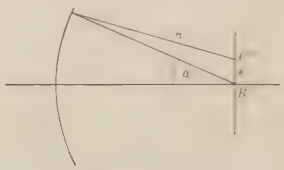

Figure 6.

n', λ_0 where n' is the refractive index of the image space, and λ_0 is the wave-length of light in a medium whose refractive index is unity*. Hence, assuming the approximate correction factor of $\cos \alpha$ as above, the displacement at C'' due to the surface element is given (in terms of the exponential convention) by

$$\lambda_0^{-1} \phi(P)\, e^{i(2\pi h' A'/\lambda)}\, dA,$$

and the total effect produced by a region of uniform amplitude and phase $\phi(P)$ over a range of numerical aperture A_y to A_z is

$$\text{displacement} = \frac{\phi(P)}{\lambda} \int_{A_y}^{A_z} e^{i(2\pi h' A'/\lambda)}\, dA' = \frac{\phi(P)}{2\pi i h'} \left(e^{i 2\pi h' A_z/\lambda} - e^{i 2\pi h' A_y/\lambda}\right),$$

$$= \frac{\phi(P)}{\pi h'} e^{\pi i h'(A_y + A_z)/\lambda} \sin \frac{\pi h'}{\lambda} (A_z - A_y).$$

In order to apply this expression it is necessary to calculate the limits A_z and A_y of the numerical aperture corresponding to the diffraction maxima on the image side. It was shown in the previous paper that the measure of obliquity of the diffracted light was given by

$$m = \frac{\text{numerical aperture of the diffracted beam}}{\text{numerical aperture of the condenser}}.$$

x' Let x' be the interval in the final image plane corresponding to the interval x in the object plane; the optical sine relation gives

$$x A_\theta = x' A'_\theta,$$

* All the expressions used will now contain the wave-length if at all as λ, the wave-length in a medium of unit refractive index. The suffix o will be omitted from this point.

where A'_θ is the numerical aperture of the converging beam corresponding to the numerical aperture A_θ of the diffracted light. The limits of the rth-order maximum are from A'_θ r

$$m + s_r = +1 \text{ to } m + s_r = -1,$$

i.e.
$$\frac{A_\theta}{A_c} + \frac{2\pi r}{v} = +1 \text{ to } \frac{A_\theta}{A_c} + \frac{2\pi r}{v} = -1,$$

or
$$A_\theta = (\lambda v/2\pi x)(1 - 2\pi r/v) \text{ to } A_\theta = -(\lambda v/2\pi x)(1 + 2\pi r/v).$$

Hence the limits for integration are

$$A_z = (\lambda v/2\pi x')(1 - 2\pi r/v) \text{ to } A_y = -(\lambda v/2\pi x')(1 + 2\pi r/v),$$

so that
$$A_z + A_y = -2\lambda r/x', \text{ and } A_z - A_y = \lambda v/\pi x'.$$

The amplitude and phase $\phi(P)$ corresponding to the rth-order term of the series

$$(\pi/v)(1^* + \Sigma e^{is_r u^*})$$

are π/v and $2\pi ru/v$.

This term therefore corresponds to a displacement represented by

$$\text{disp}_r = \frac{1}{h'v} \sin \frac{h'v}{x'} e^{2ir\gamma},$$

where
$$\gamma = \pi\left(\frac{u}{v} - \frac{h'}{x'}\right). \qquad \gamma$$

The full series then becomes

$$\text{disp} = \frac{1}{h'v} \sin \frac{h'v}{x'} [1 + e^{2i\gamma} - e^{-2i\gamma} + \ldots],$$

$$= \frac{1}{h'v} \sin \frac{h'v}{x'} [1 + 2\cos 2\gamma + 2\cos 4\gamma + \ldots].$$

Since the spacing x' of the geometrical image elements is proportional to the distance (l', say) of the image from the posterior principal focus, i.e. l'

$$x' = l'/K,$$

where K is a constant, the expression above can be written in the form K

$$\text{disp} = \frac{K}{l'} \cdot \frac{\sin(h'v/x')}{(h'v/x')} \{1 + 2\cos 2\gamma + \text{etc.}\}.$$

This expression is similar to the one obtained previously on page 203 of the original paper, but it can be more readily interpreted. When an indefinitely great number of terms are included, it is a discontinuous function having zero values except at points defined by the conditions

$$\gamma = 0, \text{ or } \pm\pi, \text{ or } \pm 2\pi, \text{ etc.},$$

i.e. where
$$h' = x'(u/v), \text{ or } h' = x'(u/v) \pm x', \text{ or etc.}$$

When a smaller number of terms are taken, the separation of the main maxima (see below on next page) is also x'; the similarity between object and image increases as more and more diffraction maxima come into action. Note that x' is the spacing prescribed by the optical sine relation for the case in question.

The maximum found where $h' = x'u/v$ corresponds to the aperture lying at the point defined by the phase angle u. As h is the distance in the object plane corresponding to the distance of this aperture from the centre of the illuminating concentration given by the condenser, the value of u is

$$u = 2\pi h A_c/\lambda.$$

Since $v = (2\pi x A_c/\lambda)$ we have

$$h' = h\,(x'/x).$$

It is therefore seen that if we move the illuminating disc the images will remain stationary in the same positions. Hence the positions will be constant for a multiplicity of illuminating points, i.e. for a source of finite size.

In order to study the effect of varying the position of the illuminating point in respect to the object, it is more convenient now to take a reference point fixed with respect to the grating; let it be some one aperture, and let the relative position of the maximum of illumination in the object plane be expressed (in terms of phase as above) by u_1. The grating space (in terms of phase) is v as before. Thus the phase interval corresponding to the first aperture beyond the illumination point will be $v - u_1$. Hence we may put $v - u_1$ in place of u in the foregoing work.

Further, the displacement denoted by h' in the image is one measured from the centre of the spherical reference surface corresponding to the illumination point, so that if displacements in the image plane are now to be measured from the new reference point corresponding to the selected aperture above, and a distance so measured is H,

$$h' = H - x'\,(u_1/v),$$

since the displacement of the image point corresponding to the central maximum must be $x'\,(u_1/v)$.

On making these substitutions γ becomes $\pi\,(1 - H/x')$ and the final series above becomes

$$\text{disp} = l'^{-1}\{(Hv/x') - u_1\}^{-1} K \sin\{(Hv/x') - u_1\}\{1 + 2\cos 2\pi\,(1 - H/x') + 2\cos 4\pi\,(1 - H/x') + \text{etc.}\},$$

which, if an infinite number of terms are included, has zero values except where

$$2\pi\,(1 - H/x') = 0, \pm 2\pi, \pm 4\pi, \text{ etc.},$$

i.e. where

$$H = \pm x', 0, \pm 2x', \text{ etc.}$$

As in the previous paper, the intensity is shown as proportional to the square of the amplitude.

$$\text{Intensity} \propto \text{Amplitude}^2 = K^2 l'^{-2}\{(Hv/x') - u_1\}^{-2} \sin^2\{(Hv/x') - u_1\}\{\text{cosine series}\}^2.$$

The intensity corresponding to a finite extension of the source of light will be found by integrating this expression between values of u_1 corresponding to the range of the image of the source projected by the condenser. An imaginary source of infinite extent will give an intensity proportional to

$$\frac{K^2}{l'^2}\{\text{cosine series}\}^2 \int_{-\infty}^{+\infty} \frac{\sin^2(V - u_1)}{(V - u_1)^2}\,du_1,$$

where V is written for Hv/x'. The value of the integral part is π, so that the intensity is proportional to V

$$(\pi K^2/l'^2)\{1 + 2\cos 2\pi(1 - H/x') + \text{etc.}\}^2,$$

and the intensities of all the image points would be equal.

Similar conclusions apply to the positions of the maxima in the interferences shown when only the central and the two first-order diffraction maxima produce the interference effects in the region of the image plane. In that case the amplitude variation is represented by the terms

$$1 + 2\cos 2\pi(1 - H/x'),$$

and has main maxima of amplitude 3, where

$$H = 0, \pm x', \pm 2x', \text{ etc.},$$

and secondary maxima of amplitude -1, where

$$H = \frac{x'}{2}, \frac{3x'}{2}, \frac{5x'}{2}, \text{ etc.}$$

The corresponding intensities are 9 and 1. The positions of these maxima will be constant for all illuminating points in a source of finite size.

§ 3. CASE OF DARK-GROUND ILLUMINATION

We may consider the case of dark-ground illumination as a reduction form of the case above. The light from the central maximum fails to reach the image, and the effects may in a likely case be due to symmetrical parts of the two first-order maxima, the second-order lying outside the aperture range of the objective. It is instructive first to consider this condition in detail. The limits of numerical aperture occupied by the first-order maxima on one side are

$$\{(\pi/p) - A_1\} \quad \text{and} \quad \{(\pi/p) - A_2\},$$

where A_1 and A_2 are the limiting numerical apertures of the dark-ground illuminator. (That is unless some parts of *these* maxima happen to fall outside the range of the objective, a condition which would need separate consideration.) In the present instance, the amplitude is found by taking the integration over the limits A_1, A_2

$$\pm\frac{x}{x'}\left(A_1 - \frac{\pi}{p}\right) = \pm\frac{A_1 x - \lambda}{x'},$$

and

$$\pm\frac{x}{x'}\left(A_2 - \frac{\pi}{p}\right) = \pm\frac{A_2 x - \lambda}{x'},$$

for the terms $r = \pm 1$. The resulting expression on writing

$$\alpha_1 = \frac{\pi h'(A_1 x - \lambda)}{\lambda x'}, \quad \alpha_2 = \frac{\pi h'(A_2 x - \lambda)}{\lambda x'}, \quad \beta = \frac{\pi u}{v},$$

is

$$(2ih'v)^{-1}\left[e^{2i\beta}\left(e^{2i\alpha_1} - e^{2i\alpha_2}\right) + e^{-2i\beta}\left(e^{-2i\alpha_1} - e^{-2i\alpha_2}\right)\right],$$

or

$$2(ih'v)^{-1}\sin(2\beta + \alpha_1 + \alpha_2)\sin(\alpha_2 - \alpha_1).$$

With the same substitutions as before, viz. $u = v - u_1$, $h' = H - x'u_1/v$, the expression becomes

$$\frac{2}{i(Hv - x'u_1)} \sin 2\pi \left\{ 1 - \frac{H}{x'} + \frac{x}{2\lambda}\left(\frac{H}{x'} - \frac{u_1}{v}\right)(A_1 + A_2) \right\} \sin \pi \left\{ \frac{x}{\lambda}\left(\frac{H}{x'} - \frac{u_1}{v}\right)(A_2 - A_1) \right\}.$$

It is to be remembered that this expression only relates so far to the case of a single illuminating point. The extreme factors represent a broad amplitude-variation of the $U^{-1} \sin U$ type, in which the spacing of the bands would depend on $A_2 - A_1$, and would become broader as $A_2 - A_1$ diminishes numerically.

The remaining factor is of such a form that the value of H giving the maximum value of the sine term must depend on u_1, i.e. on the relative location of the central point of illumination in the object plane. Hence any multiplicity of illuminating points would be bound to yield a set of non-coincident maxima, and as we must in practice always use a source of light having finite dimensions, the interferences will not be visible in ordinary practice. They could not be seen in any case unless the range of the illuminating points were small in comparison with the object spacing.

The next case for consideration is that in which the aperture range of the objective is great enough to allow both first- and second-order maxima to take effect, the direct-light maximum of zero order still being screened. The presence of the second order is represented by adding the appropriate term to the above expression. The numerical-aperture range of the second-order maximum on one side is

$$\{(2\pi/p) - A_2\} \quad \text{to} \quad \{(2\pi/p) - A_1\}.$$

The second-order term for one side will therefore be

$$(2ivh)^{-1} e^{4i\beta} (e^{2i\alpha_1'} - e^{2i\alpha_2'})$$

in addition to the corresponding first-order term

$$(2ivh)^{-1} e^{2i\beta} (e^{2i\alpha_1} - e^{2i\alpha_2}),$$

α_1' where $$\alpha_1' = \frac{\pi h'(A_1 x - 2\lambda)}{\lambda x'},$$

α_2' $$\alpha_2' = \frac{\pi h'(A_2 x - 2\lambda)}{\lambda x'}.$$

When symmetrical dark-ground illumination is employed there will be in addition the corresponding terms obtained from the above by changing the signs of all the exponential terms in the two brackets. Adding the two expressions set out above we obtain

$$(2ivh')^{-1} e^{2i\beta} (e^{2i\alpha_1} - e^{2i\alpha_2}) (1 + e^{2i\gamma}),$$

as might have been found at once from the general formula of page 239. This can be expressed in the form

$$(vh')^{-1} e^{i(2\beta+\alpha_1+\alpha_2)} (1 + e^{2i\gamma}) \sin(\alpha_1 - \alpha_2),$$

or

$$2(vh')^{-1} e^{i(2\beta+\gamma+\alpha_1+\alpha_2)} \sin(\alpha_1 - \alpha_2) \cos\gamma.$$

When the illumination is symmetrical the combined result is clearly

$$4(ivh')^{-1} \sin(2\beta + \gamma + \alpha_1 + \alpha_2) \sin(\alpha_2 - \alpha_1) \cos\gamma,$$

or, written in full,

$$4\,(ivh')^{-1} \sin\left\{3\pi\left(\frac{u}{v}-\frac{h'}{x'}\right)+\frac{\pi h'x}{\lambda x'}\,(A_1+A_2)\right\} \sin\frac{\pi h'x}{\lambda x'}\,(A_1-A_2)\cos\pi\left(\frac{u}{v}-\frac{h'}{x'}\right).$$

On substitution for u and h' as before, the first sine term becomes

$$\sin\left\{3\pi\left(1-\frac{H}{x'}\right)+\frac{\pi x}{\lambda}\,(A_1+A_2)\left(\frac{H}{x'}-\frac{u_1}{v}\right)\right\},$$

and this clearly has its maxima at values of H depending on u_1. The same statement applies also to the second sine term, but *not* to the cosine term. Hence the effects of the cosine maxima will be cumulative as the dimensions of the source of illumination are increased, and these maxima occur at consecutive values of H differing by x' as shown above, so that they correspond to the intervals of the geometrical image points.

When further diffraction maxima come into action, the expression in the symmetrical case may be written

$$(vh')^{-1} \sin(\alpha_1-\alpha_2)\,[e^{i(2\beta+\alpha_1+\alpha_2)}\{1+e^{2i\gamma}+e^{4i\gamma}+\text{etc.}\} - e^{-i(2\beta+\alpha_1+\alpha_2)}\{1+e^{-2i\gamma}+e^{-4i\gamma}+\text{etc.}\}],$$

$$= 2\,(ivh')^{-1} \sin(\alpha_2-\alpha_1)\,[\sin\delta+\sin(\delta+2\gamma)+\sin(\delta+4\gamma)+\text{etc.}],$$

where $$\delta = 2\beta+\alpha_1+\alpha_2.$$

On expanding the sine brackets, the expression can be written

$$2\,(ivh')^{-1} \sin(\alpha_2-\alpha_1)\,[\sin\delta\,(1+\cos 2\gamma+\cos 4\gamma+\text{etc.}) + \cos\delta\,(\sin 2\gamma+\sin 4\gamma+\text{etc.})].$$

The number of terms to be taken in the cosine and sine series depends on the number of diffraction maxima which can contribute their effects. The case in which the first and second orders above are present has been dealt with above. If a considerable number are acting, the cosine series tends to give small values except at the points representing the geometrical image points, as has been shown, but the effect of the sine series is not cumulative in the same way, and its effect is negligible in comparison with that of the cosine series. Thus as the additional maxima come into play, there would be a closer and closer approximation to a series of extremely localized images representing the series of indefinitely small apertures constituting the object.

There is no need to extend the above treatment to the case of a series of apertures of finite width, since there is no doubt that the conclusions would be very similar.

It is therefore clear that the first-order maxima cannot by themselves produce interference phenomena representing the geometrical image in any exact sense, and that it is only when maxima of different orders are allowed to act more or less in entirety that the interferences in a definite surface bear a geometrical relation to the object.

The treatment of this subject on the lines of the Abbe principle (following up the

work begun by Cross) will not be given here. The essential features of the discussion would follow lines very similar to those used in the theory of bright-ground images. It would be found that the interferences due to pairs or groups of homologous maxima would agree only in the geometrical image. In dark-ground illumination the direct beams, on either side of the normal, would be considered independently; this seemed to the writer to be a matter requiring closer examination, but the results of this paper, so far as they go, constitute a justification of the theoretical applicability of the Abbe principle. Of course the conditions assumed in the theory are greatly modified in practice. It is not claimed that the use of a symmetrical dark-ground illuminator does always produce an accurately focused image of the light-source in the object plane. The argument is rather that even if this were the case, the necessity of considering the interference phenomena in image-formation cannot be avoided. It seems worth while again to point out that when the object consists of a transparent structure, the interference effects on which we rely to produce a picture can only be made effective by a deliberate restriction of the apertures of illumination and observation, with a consequent uncertainty as to the precise geometrical configuration of the object.

§ 4. PRACTICAL TESTS

In order to gain some experience of actual effects with an object of known character under dark-ground illumination, a few trials have been carried out with a set of Grayson's test-rulings. The specimen, kindly supplied by Sir Thomas Lyle, contained a series of bands ruled in a realgar film. The rulings are protected by a cover-glass, and there is presumably a film of water or some other medium between the rulings and the cover-glass to allow of the passage of light at numerical apertures greater than unity. The bands consist of a few lines at 5000, 10,000, 15,000, and so on up to 60,000 lines to the inch.

The objective was an excellent 16-mm. apochromat by Beck; the full numerical aperture was 0·35, but it could be limited to 0·18, if required, by means of a diaphragm placed near the posterior focus. The eye-piece used for visual trials was a × 20 compensating.

The illumination was produced by a 100-c.-p. pointolite arc placed at the focus of an achromatic auxiliary condenser which projected a parallel beam into the substage condenser of the microscope. This was a well-made condenser of the Abbe type with an iris diaphragm; it was fitted with three special diaphragms having annular apertures corresponding to the following measured ranges of numerical aperture: 0·25 to 0·34; 0·375 to 0·45; 0·6 to 0·7. The centering of the illumination received due care.

Visual tests showed that the objective worked well at the normal tube-length; the system is not very sensitive to changes of tube-length at low numerical aperture.

A series of photographs was taken as follows, after some preliminary trials to ascertain correct exposures.

Set A. Objective numerical aperture = 0·18. *Blue filter, Wratten No.* 45 *used throughout.*

1. Bright-ground illumination. Full cone (condenser aperture 0·18). Illuminant focused in object plane.

2. Same conditions as in 1, but illuminant no longer in focus. Objective aperture still filled with light.

3. Dark-ground stop no. 1. Illuminant in focus.

4. Dark-ground stop no. 2. Illuminant in focus.

5. Dark-ground stop no. 3. Illuminant in focus.

Set B. Objective numerical aperture = 0·35.

1. Bright-ground illumination. Full cone (condenser aperture 0·36). Image of illuminant in focus. Blue filter.

2. Dark-ground stop no. 2. Illuminant in focus. Blue filter.

3. Dark-ground stop no. 2. Illuminant in focus. Red filter, Wratten No. 1. Panchromatic plate used for photography.

Examination of the plates gave for set A the results shown in table 1.

Table 1

Set A

	Objective numerical aperture	Illumination numerical aperture	Notes
1	0·18	0·18 Bright ground	Bands resolved up to 15,000. None higher
2	,,	,,	Same as with no. 1
3	,,	0·25 to 0·34 Dark ground	Bands resolved up to 10,000. None higher
4	,,	0·375 to 0·45 Dark ground	Bands resolved up to 15,000
5	,,	0·6 to 0·7 Dark ground	Bands resolved up to 20,000

The above results are largely in accordance with the predictions of elementary theory*; it will be noticed that the introduction of dark-ground illumination gives a smaller resolving-power than with bright-ground until the numerical aperture of the dark-ground illumination is practically three times that of the objective. In the case of no. 5, where the dark-ground illumination has a numerical aperture still higher, the images of the 15,000 and 20,000 bands no longer consist of equally spaced parallel lines, but have a somewhat irregular spacing. The correct number of

* See L. C. Martin, *An Introduction to Applied Optics*, 2, 127 (Pitman, 1932). Elementary theory indicates that if the higher limiting aperture A_i of the dark-ground illumination is only slightly greater than that of the objective with aperture A_0, the minimum grating-interval that can be resolved is little better than λ/A_0, i.e. the interval must be about double that resolvable by the same objective with bright-ground illumination. As the aperture of the dark-ground illumination increases, the minimum interval resolvable is $2\lambda/(A_0 + A_i)$, so that when $A_i = 3A_0$ the minimum interval resolvable is about $\lambda/2A_0$; the performance in this respect is then as good as can be obtained with the same objective with bright-ground illumination. The best that can be expected of an apochromatic objective in numbers of lines per inch visually resolvable is approximately 10^5 times the numerical aperture. Thus at 0·18 we may hope to resolve 18,000 lines per inch, and so on.

lines, as shown by bright-ground photographs, appear, but the irregularity seems to be due to a diffraction phenomenon of the nature of an end effect.

The results for set B are shown in table 2.

Table 2

Set B

	Numerical aperture of objective	Numerical aperture of illumination	Notes
1	0·35	0·36 Bright ground	Bands partly resolved at 35,000. None higher
2	,,	0·375 to 0·45 Dark ground, Blue light	Lines of some kind visible both in 30,000 and 35,000 bands, but 25,000 band not well resolved. 20,000 well resolved
3	,,	0·375 to 0·45* Dark ground, Red light	Lines of some kind visible up to 30,000. None higher

In this set of photographs it was a surprise to find an apparent resolution of the lines in the 30,000 and 35,000 bands, seemingly in contradiction to the predictions of theory. It would not be expected that the 35,000 band would be resolved unless the aperture of the dark-ground illumination were increased to a value exceeding unity in the present case. The resolution is less marked with red light, but is, at first sight, still in excess of expectation†.

The relative spacing of the lines seemed to be so much in accord with what might be expected from the known frequencies, that the appearances were actually mistaken for real images until the plates were measured, with the results shown in table 3. There are slight differences of magnification between the plates, but the numbers in the rows should nearly agree if genuine images are produced.

The spurious lines are diffraction fringes associated with the edge effects at the sides of the bands. Their frequency is evidently dependent on the angle of diffraction of the first-order maximum, and hence there is a very deceptive variation roughly corresponding with the frequency of the actual rulings.

In order further to elucidate the conditions arising in the above cases, an examination of the diffraction maxima in the posterior focal surface of the objective was carried out; first for the condition obtaining for plate B 2. Each band of rulings produces its own set of diffraction maxima; hence a Ramsden eye-piece was employed and a slit diaphragm was placed in its focal plane in the microscope so as to expose the image of one band at a time. The corresponding diffraction maxima could then be observed in the exit pupil of the eye-piece without interference from those of other bands. The results are indicated in figure 7.

* As the condenser is non-achromatic, the actual numerical apertures obtained with blue and with red light may differ slightly.

† Enlargements of the negatives were prepared with a view to reproduction, but it was found that the detail would in all probability be lost in printing and the project was abandoned.

Table 3. Mean Interval (arbitrary units)

Band	Plate B 1 Bright ground, blue	Plate B 2 Dark ground, blue	Plate B 3 Dark ground, red	Plate C 1 Dark ground unilateral, blue
15,000	Out of focus	0·981 G 9 lines (distinct)	1·003 G 9 lines (distinct)	0·975 G 9 lines (distinct)
20,000 11 lines	0·756 G 11 lines (distinct)	0·736 G 11 lines (distinct)	0·937 S Not more than 9 lines visible (indistinct)	0·744 G 11 lines (fairly distinct)
25,000 14 lines	0·602 G 14 lines (distinct)	0·783 S Only 10 lines (some indistinct)	0·694 S Only 12 lines (indistinct)	0·701 S (Only distinct over part of band)
30,000 17 lines	0·470 G 17 lines (distinct)	0·610 S Not more than 13 lines (some indistinct)	0·593 S (Only distinct over part of band)	0·567 S Not more than 14 lines visible
35,000 19 lines	0·433 G 19 lines (faint)	0·508 S (Lines visible only over half the band)	(No lines visible)	0·504 S (Lines visible only over part of band)

G indicates a genuine image. S indicates a spurious image.

The maxima would, if the objective were large enough, be complete rings, of which only limited parts are exposed in practice. Figure 8 is intended to indicate the approximate distribution of the complete maxima in the case of the 20,000 band, for blue light.

Figure 7.

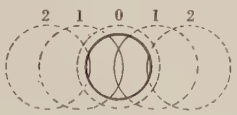

Figure 8.

With a greater number of lines, or with a greater wave-length, the distance between the maxima increases and the second-order maxima are no longer included within the objective aperture. Resolution can therefore no longer be obtained with dark-ground conditions.

Since it seemed possible that the presence of two first-order maxima might be responsible for some spurious fringes at least, a photograph was made with unilateral dark-ground illumination, so that only one first-order maximum should be exposed. In order to do this the annular aperture of the dark-ground diaphragm was closed except for a segment at one extremity of a diameter perpendicular to the lines. Blue light was used, and the results of this plate, C 1, are given in table 3. The results were much the same as before with the symmetrical illumination of the same numerical aperture, except that the spurious fringes are markedly one-sided.

It must not be inferred from this that symmetrical illumination is not superior to unilateral illumination in ordinary microscopy. The resolution of rulings is a very special case.

Using the 20,000 band alone, photographs were made of the exit pupil with unilateral illumination, both in blue light and in red light, figure 9. It will be noticed that there are secondary maxima present in both cases, but with the red there is only one main maximum, the first-order, present, as against both first and second-order maxima with the blue. This then is in accordance with the well-known criterion for distinguishing between real and spurious resolutions.

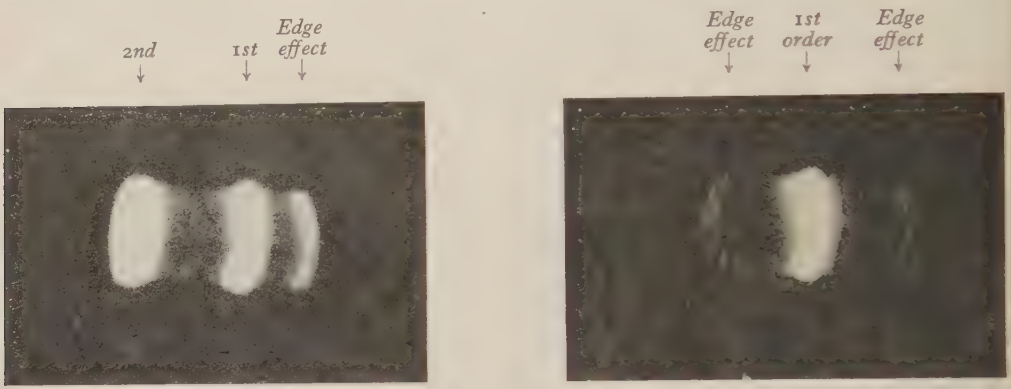

Figure 9. The exit pupil of the microscope with one-sided dark-ground illumination on the 20,000 band.

The effects of the secondary maxima, which must be fairly prominent when the ruling consists of a few lines only, are well worth investigation, and the theory of the edge effects has not been investigated, so far as the writer's knowledge goes. It is therefore hoped to deal with these problems in the near future, as they are of considerable practical importance in the interpretation of the image in the microscope.

Finally it is not pretended that the above discussion or experiments convey an adequate account of the *general* question of the trustworthiness of the image in dark-ground illumination; nevertheless the theoretical and practical enquiries must start with the simplest possible cases. This paper should, however, convey a needed note of warning, as claims have been made that symmetrical dark-ground illumination is adequate to realize in every case the full resolving power of the objective. These claims cannot be theoretically substantiated. On the other hand the method of dark-ground observation may, by increase of contrast, reveal objects and structures which cannot be observed in any other way.

DISCUSSION

Mr T. Smith. We should all agree with the author that an adequate theoretical discussion of microscopic imagery presents very great difficulties. Even the limitation to two dimensions of itself introduces some troubles of a formal nature. Another difficulty is to know how far we may apply results obtained, as in the present paper, by assuming that the numerical aperture is small. By way of illustration of these difficulties we may propound the question whether the ordinary expression for the numerical aperture is universally applicable, so that for instance any two portions into which a sphere is divided by a plane necessarily have the same numerical aperture, and in particular, whether the numerical aperture of a complete sphere is zero, and whether we must include among "small" apertures a nearly complete sphere.

Putting on one side the troubles connected with the range over which it is permissible to apply the theory in its present state, the difficulties which remain are by no means slight. In order to obtain results which may be stated in definite terms, it becomes necessary to sum the series which present themselves to infinity, and in so doing to assume conditions which depart to some extent from those which represent the actual problem. For we cannot of course in fact pass through a microscope objective light proceeding in a given direction from each of a very great number of parallel apertures. Perhaps it may be possible to show that when any considerable number of terms is taken the results will not differ very significantly from those corresponding to an infinite number of terms: if this were done it would add much to the strength of the arguments used. As things are I am inclined to agree with the author's conclusions for small apertures rather because they are in general concordance with my expectations than because the theory shows that in practice these conclusions must hold. If these theoretical difficulties can be surmounted I am sure the result will repay the labour involved. At the moment I believe that the most direct way to resolve at least one of the problems which the author has in mind would be to apply the experimental process he has described in his fourth section to the particular cases in which it is thought that exactly corresponding results have been obtained with dark- and with bright-ground illumination. To be of value the dark-ground conditions would have to be precisely those which are claimed to give these results, and therefore they must be set up by those who make these claims, and the counts and measurements made without disturbing these conditions. The bright-ground conditions present of course an altogether simpler question, and need occasion no difficulty.

I should be glad if the author could see his way to adding a fuller explanation to figures 7 and 8, as I am a little puzzled by the circles with unidirectional illumination of a plane grating. Also it would be of interest if he recorded on p. 231 what he includes in referring to the Abbe principle, since he clearly implies more than a treatment which starts out in the way described in the last ten lines of p. 233.

It will be clear from these comments that I think there is still needed a great deal of work, both theoretical and experimental, before we can understand the

high-aperture microscope and its limitations fully. The subject is unquestionably of great difficulty, and we should all feel grateful for any attempts that are made to add to our understanding and to face these difficulties openly. At the risk perhaps of appearing unduly critical, I have pointed to some difficulties which I think deserve attention in this paper, and I shall personally feel very grateful to Dr Martin if he can now or on some future occasion present the argument in a form which removes them. In any event I am sure his paper will be read and welcomed by those who are interested in microscopy and optical instruments generally.

Author's reply. It is by no means easy to determine how far the results of an elementary two-dimensional discussion can be applied in practice, yet it seems fair to conclude that similar methods of analysis could be applied, and, as Mr Smith remarks, the fact that the elementary theoretical results are in harmony with actual observations should support the view that we are somewhat nearer an adequate theoretical approach than was previously the case. If I understand the matter rightly, the scope of the present treatment should cover a wider range of numerical apertures than would be comprehended in a purely paraxial type of case.

It is quite true that in dealing with the problem of the row of apertures the analysis can only be carried through to an algebraical solution by assuming an infinite number of apertures. Let us, however, consider a typical microscopical case in which some object with, say, 30,000 lines or dots per inch is being examined with an 8-mm. microscope objective of numerical aperture 0·65. The front lens of such an objective will have a fairly large actual diameter (approaching 1 cm. in some cases) and the field of a medium-power eye-piece may include a 1-mm. range of the object. It is therefore by no means unfair to assume that we have at least a thousand apertures all of which can contribute to the first-order and second-order maxima at the back of the objective. This represents, in our case, *five hundred terms* of a series like

$$\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \text{etc.},$$

and it does not require a great deal of argument to see that the sum to infinity is a close approximation to the result. The expression near the foot of p. 237,

$$(\pi/v)\,(1^* + \Sigma e^{is_r u^*}),$$

depends on an approximation of the same order. Even Mr Smith has sometimes been satisfied by fewer terms than five hundred in some aberration series! I know, of course, that this one is only slowly convergent, relatively speaking.

In another case I obtain the expression for the amplitude in the final image (near the foot of p. 239) as a series in which the successive cosine terms represent the successive diffraction maxima, and we can calculate the result of two, three, or more terms as more maxima are included in the range of numerical apertures. The sum to infinity is mentioned to imply that the distribution of light in the image only approximates closely to that of the object when a relatively great numerical aperture is allowed; it is not a step in the course of the argument.

With regard to figures 7 and 8, it will be remembered that the aperture of the dark-ground illuminator is restricted to a fairly narrow annular ring, and I have

tried to show the positions of the corresponding annular diffraction maxima by broken-line circles. The thick circle in figure 8 represents the limit of the objective, and circle 0 suggests the diffraction maximum of zero order (the direct light); circles 1 and 2 suggest the first-order and second-order maxima on each side. Thus figure 8 explains the case on the extreme left of figure 7. If the diffraction is effected by rulings of greater and greater frequency the second-order maxima pass quite out of the field, and leave the first order only, as is suggested in the successive parts of figure 7. In the case of the use of one-sided illumination, a segment of the annular aperture of the illuminator is exposed, and the corresponding diffraction maxima are similarly segmental.

I am also asked what is included in the Abbe principle? I understand it to mean the analysis of the method of image formation by the discussion of the diffraction of plane wave-trains at the object; it is usually assumed that the actual illumination can be resolved into groups of plane wave-trains incident at various angles. The mathematical foundations of this assumption have been examined by Dr Johnstone Stoney* and the applications of the principle have been worked out in papers too numerous to mention, but grave doubts have been felt as to whether the principle is valid in conditions of critical illumination.

I feel fairly confident that the principle is definitely applicable in the case of the type of object assumed in this theory, i.e. apertures regularly spaced in a thin opaque screen. On the other hand, we have the assurance of distinguished microscopists like Mr Conrad Beck that he has obtained resolutions in dark-ground illumination, when observing regular structures like diatoms, which are beyond the predictions of theory, and Mr J. Smiles tells me that he and Mr Barnard have been able to photograph images giving more than the theoretical resolution even when the diminished wave-length is taken into the theoretical account in dark-ground illumination with an ultra-violet dark-ground illuminator. I agree with Mr Smith that some such trials as he suggests might well be made, because if these claims were justified in practice it would point to some special physical explanation such as the partial decoherence of the light vibrations produced by the elements of the structure, or to the necessity of confessing that the ordinary methods of analysis are hopelessly at fault in the problem of resolution by high powers.

Mr Smith's problems about the numerical aperture of a complete sphere, etc., do not seem to arise in ordinary microscopical practice, in which we deal with objects illuminated by transmitted light alone, or by reflected light alone. Image formation due to both reflected and transmitted light would have to be analysed with respect to the two actions, and the numerical aperture in either case could not exceed the value of the relevant refractive index. I do not see that there is any difficulty arising through such questions in the present connection.

Finally I should like to thank Mr Smith for his criticism and for some suggestions, communicated privately, which have helped me considerably to improve the formal presentation of some mathematical parts of the paper.

* *Phil. Mag.* October, November and December, 1896.

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ON THE USE OF CHARCOAL IN MAINTAINING HIGH VACUA

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ABSTRACT. Measurements have been made of the limiting pressures reached by charcoals and silica gels cooled with liquid air or liquid hydrogen when a small, constant stream of gas is admitted into the apparatus.

§ 1. INTRODUCTION

In experimental work it is often essential to maintain a low pressure in a metal box with waxed joints. Since the metal is continually giving off gas, the box must be continuously evacuated and this is usually done with a mercury or oil condensation pump connected to the box by as short a length of wide tube as is practicable. The use of these pumps very often gives trouble, since it is impossible to avoid severe vibration of the apparatus, nor is it always convenient to place a fast pump sufficiently near the apparatus to be exhausted.

It is the purpose of these experiments to find out to what extent the older method of using cooled charcoal can be employed to maintain a low pressure in a vessel continually giving off small quantities of gas. Although many measurements have been made[1] on the limiting pressures reached by well-exhausted charcoals and liquid air when sufficient time is allowed for complete absorption, very few determinations appear to have been made on the equilibrium pressures reached when a constant stream of gas is introduced into a vessel containing cooled charcoal.

In the experiments to be described the pressures were measured with an ionization gauge. This type of gauge was chosen because it is simple to make and can be made to read the lowest pressures. The chief disadvantage of this type of gauge, namely, that it is difficult to degas sufficiently, was in these measurements an advantage, the hot metal of the gauge being used as the supply of gas of composition suitable for the experiment.

The following materials were used as absorbers: commercial silica gel; a special silica gel, completely freed from adsorbed salts by prolonged dialysis[2]; birchwood charcoal*; gas-mask charcoal Rb I†; Norit AK T II†; charcoal G 1000†; specially prepared charcoal used by Dr Lambert.

The experimental procedure was to place the charcoal or gel in a wide vertical glass tube with a side piece leading through a wide-bore tap to the pumping system.

* Obtained from Messrs Hopkins and Williams, London.

† Samples of these charcoals were kindly presented by the British Carbo Union, Ltd.

The ionization gauge was attached vertically above the charcoal by means of a large ground joint lubricated with Shell Apiezon L grease of negligible vapour pressure. Before the charcoal was introduced into the apparatus it was roughly evacuated with an oil pump, while the silica gel was heated in an open silica dish over a large Meker burner. After this treatment a weighed quantity (about 8 gm.) was introduced through a funnel into the vertical tube, and maintained at from 350 to 400° C. with the pump running until the evolution of gas became very slow; this usually took about 3 hours. No attempt was made to remove the last traces of gas by prolonged evacuation.

§ 2. PRESSURE-MEASUREMENT

Two series of experiments were carried out. In the first a 2-stage mercury condensation pump was used to evacuate the system, and the ionization gauge consisted of a cylindrical anode 1 cm. in diameter, a concentric spiral grid 0·4 cm. in diameter, and a central filament. Under working conditions the anode voltage was 1000 V., the anode current 6 mA. and the grid voltages − 20 to − 30 V. The filament current was supplied by a small transformer and the grid current was measured with a reflecting galvanometer.

The measure of the pressure in such a gauge is the ratio R of grid current to plate current. It is easy to calculate approximately, from the dimensions of the gauge, the numerical factor connecting the measured quantity R with the pressure. R

The electrons in passing from the filament to the plate are accelerated from rest to 1000 V., producing a certain number of positive ions by collision with the gas in the gauge. These are collected by the negative grid, and constitute the grid current. Now it is known experimentally[3] that a 1000-volt electron on being brought to rest in air at 1 mm. pressure, has a path of 5 cm. and produces 21 positive ions. Thus, if p is the pressure in the gauge in mm. of mercury, $R = \frac{21}{5} \times \frac{1}{2}p$, or $p = \frac{1}{2}R$ approximately. p

Since this gauge was not sufficiently sensitive to measure the lowest pressures reached, for the second series of experiments another gauge was made up consisting of a fine wire anode between two flat plates to collect the positive ions. This arrangement of the electrodes produces a more sensitive gauge, since the electrons circulate several times round the wire anode and so have a longer path in the gas. A suitable gauge is described by Jaycox and Weinhardt[4], who have also calibrated it against a McLeod gauge. The same calibration was used.

§ 3. EXPERIMENTAL RESULTS

Series I. The volume of the apparatus was 190 cm.^{3}; the whole was baked out and the rate of rise of pressure due to gas coming from the gauge was measured. As the mean of six readings the pressure rose from $1{\cdot}2 . 10^{-4}$ to $1{\cdot}1 . 10^{-3}$ mm. in 40 seconds, corresponding to 475 cm.^{3} of gas per second at 10^{-5} mm.

On connecting to the pump the pressure fell to $1{\cdot}1 . 10^{-4}$ mm., corresponding to a pumping speed of 43 cm.^{3}/sec. This was in good agreement with Knudsen's

formula for the resistance of a tube system, according to which the maximum possible speed was 80 cm^3/sec.

Table 1. Series I

Adsorbent material	Observations
Commercial silica gel	When the residual gases were removed by the pump, the pressure fell to $3{\cdot}1 \cdot 10^{-5}$ mm., and on disconnexion of the pump rose at first quickly and then slowly to about $2 \cdot 10^{-4}$ mm., after which it rose very slowly at about $1{\cdot}5 \cdot 10^{-6}$ mm. per minute for an indefinite period
Specially purified silica gel	With the tap turned to the pump, the pressure fell to $2{\cdot}2 \cdot 10^{-5}$ mm., and on disconnexion of the pump the pressure rose steadily for about 10 minutes to $4{\cdot}8 \cdot 10^{-5}$ mm. and then remained constant
Dr Lambert's charcoal	With the tap turned to the pump, the pressure fell to $8{\cdot}5 \cdot 10^{-6}$ mm., and on disconnexion of the pump the pressure of the residual gas rose steadily at $7 \cdot 10^{-6}$ mm. per minute

Table 1 shows that the purified gel is much better than the commercial gel, especially for absorbing the residual gases (probably hydrogen), but cannot be used to replace charcoal if pressures lower than $5 \cdot 10^{-5}$ mm. are wanted.

A comparison of the initial rate of accumulation of non-absorbed gas with the measured speed of the pump showed that for the silica gels the lowest pressure reached was determined by the vapour pressure of the absorbed gas.

Series II. The volume of the apparatus was 820 cm^3, and the rate of evolution of gas was 172 cm^3/sec. at 10^{-5} mm. When the gauge was exhausted by the pump alone the pressure fell to $1{\cdot}9 \cdot 10^{-5}$ mm., corresponding to a pumping speed of 90 cm^3/sec.

Table 2. Series II

Adsorbent material	Observations
Dr Lambert's charcoal	With both pump and charcoal the pressure fell to $4{\cdot}7 \cdot 10^{-6}$ mm., and on disconnexion of the pump only rose slightly, reaching a constant value of $7 \cdot 10^{-6}$ mm. in 5 minutes
Rb I	The lowest pressure reached was $3{\cdot}3 \cdot 10^{-5}$ mm. (i.e. higher than with the pump alone), and on disconnexion of the pump the pressure rose steadily and indefinitely
Norit AK T II	This charcoal gave off relatively small quantities of gas on evacuation but formed a small white sublimate on baking. The behaviour was very similar to that of Rb I
G 1000	This gave a limiting pressure on the pump of $1{\cdot}4 \cdot 10^{-5}$ mm., but on disconnexion of the pump the pressure rose steadily at about $4 \cdot 10^{-6}$ mm. per minute
Birchwood charcoal	The lowest pressure was $3{\cdot}3 \cdot 10^{-6}$ mm., and the residual gas produced a rise of only $5 \cdot 10^{-7}$ mm. per minute

The birchwood charcoal was also cooled with liquid hydrogen. The pressure fell to $5{\cdot}5 . 10^{-7}$ mm. and remained at this low value as long as the liquid hydrogen lasted—about 15 minutes.

§ 4. DISCUSSION OF RESULTS

A comparison of the two results for Dr Lambert's charcoal, in which the pressures were measured by independent gauges on different specimens, shows remarkably good agreement. The ratio of the rates of evolution of gas in the two experiments was 2·7 : 1, while the ratio of the pressures as measured on the two gauges was 1·8 : 1.

The experiments show that the charcoals specially prepared to have large absorptive power at room temperatures do not give satisfactory results under vacuum conditions. Even with the best charcoals the pressure rises slowly owing to the accumulation of unabsorbed gas, so that if the low pressure has to be maintained for several hours it is essential to use a pump.

The most effective procedure is to cool the exhausted charcoal with liquid hydrogen. Since the rate of evolution of gas in the gauge has been measured and the dimensions of the tube connecting the gauge to the cooled charcoal are known (length 20 cm., diameter 1·8 cm.), one can calculate the chance that a molecule of gas will be reflected on hitting the cool surface. If this is zero so that every molecule hitting the charcoal is absorbed, the calculated pressure in the gauge is $5{\cdot}0 . 10^{-7}$ mm., a result which is in excellent agreement with the observed value of $5{\cdot}5 . 10^{-7}$ mm. Thus a surface of charcoal cooled with liquid hydrogen approaches the theoretically perfect absorbing material.

§ 5. ACKNOWLEDGMENT

In conclusion I have to thank Prof. F. A. Lindemann, F.R.S., for extending to me the facilities of his laboratory, and for his kind interest in the work.

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THE ILLUMINATION-RESPONSE CHARACTERISTICS OF VACUUM PHOTOELECTRIC CELLS OF THE ELSTER-GEITEL TYPE

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ABSTRACT. The authors deal with the present status of the vacuum photocell as regards proportionality of photoelectric current to incident illumination, and give a series of observations which show that, while excellent cells are obtainable, their employment for precise work without a previous test is unsafe. A theory is developed which accounts for the observed results and gives over a limited range an equation of the form $S = K_1 + K_2/\sqrt{E}$, where S is the cell-sensitivity and E the illumination. Attention is called to the dependence of the illumination characteristic upon the colour of the light employed. The bearing of the theory on the application of Talbot's law to non-linear photocells is discussed.

§ 1. INTRODUCTION

The relation between photoelectric current and incident illumination has been the subject of a large proportion of the investigations dealing with the photoelectric effect, though the point of view from which the question has been approached has changed progressively with lapse of years. Early in the history of the subject the natural point of view was the academic one having as its object the explanation of the mechanism of the photoelectric effect. Later, with increasing realization of the possibilities of the photoelectric cell as a photometer, the practical aspect has slowly superseded the academic and it has become a matter of importance to know to what extent a given cell may depart from the ideal as regards proportionality of photoelectric current to illumination.

The classical papers by Elster and Geitel[1], Lenard[2], Griffith[3], Richtmyer[4], and Ives[5] describe most of the methods available for investigating the illumination-response curve, and bring to light many of the difficulties associated with precise measurements on these lines. They establish the law of proportionality in the general physical sense over very wide ranges of illumination, and many subsequent papers[6, 7, 8, 9, 10, 11] confirm this result for both vacuum and gas-filled cells, though in the case of the latter it was soon recognized that too high a *gas amplification*, as it is now termed, introduced measurable deviations from linearity of response[5, 12, 13, 9, 14, 15]. Long-period fatigue effects became apparent under

these conditions [14, 15] and affected the validity of the measurements. Finally a short-period effect, with influence on the performance of gas-filled cells under flickered illumination [16, 17, 18], was also found. The primary electron-emission has, however, been found to commence and terminate simultaneously with the incident illumination [19]. Among other factors affecting the shape of the response curve there are to be noted electrode-disposition and the presence of more or less isolated films of metal in the cell [12, 5, 32].

To sum up, the above-mentioned researches leave little doubt as to the proportionality of photoelectric current to illumination under ideal conditions, and give considerable insight into the various difficulties encountered in attaining strict proportionality in practice. As a result it is now possible to obtain vacuum cells in which the approximation to strict proportionality is close enough for ordinary photometric purposes, but this fact does not dispose of the necessity for carefully testing a cell intended for precision photometry. The troubles likely to be met with in the application of the photocell to accurate photometry have been summarized in papers by Ives [20], Gibson [21], Harrison [22], Ives and Kingsbury [23], and Winch [24], and in a paper recently submitted to the *Illuminating Engineer* by the present authors.

In view of the importance of the subject of linearity of response the authors have therefore carried out the tests described in the present paper with the objects (i) of assessing the usefulness of certain cells for commercial photometric work and (ii) of utilizing the results obtained for a more detailed study of the reasons for non-linearity, and of its bearing on the use of the photocell for heterochromatic photometry. The paper is limited to cells of the vacuum type, which are the only ones now used for precise measurements.

§ 2. EXPERIMENTAL METHODS

Experimental methods of investigating photocell illumination characteristics differ chiefly in the ways in which the known changes in illumination are produced. These include the application of the inverse-square law and of Beer's law, the addition of several light-sources, and the use of crossed Nicol prisms. The use of crossed Nicol prisms is attended by several difficulties and is open to the objection that polarization effects may be introduced, although some investigators have claimed high accuracy for the method [1]. Mathematically, the inverse-square law is equally sound provided that the usual precautions as to size of source and so on are taken. It is, however, necessary to employ a diffusing window in front of the photocell, or alternatively to place the cell inside an integrator illuminated through an aperture by the light-source, and either of these arrangements introduces some uncertainty as to the true illumination values. The validity of the results depends on the attainment of perfect diffusion by the window, or of a uniform coating on the interior of the integrator. Again, the use of filters with monochromatic illumination, and the application of Beer's law of absorption [1] are inapplicable if the photocell response curve obeys the relation found by Steinke [9], because in this case the departure of the response curve from linearity appears only as a change in the

exponent in Beer's relation [10]. The absolute value of this exponent is seldom known with sufficient accuracy for small deviations from the accepted value to be laid to the charge of the photoelectric measurements. The additive method, used by Elster and Geitel in 1893 [1] and dealt with again more fully by Campbell [25], remains the most suitable method and was used for all the work described in the tables given in the present paper. (The electrometer apparatus shown in figure 1 and described hereafter served only to draw attention to the influence of colour on the observed results.)

Among sources of error on the electrical side it is perhaps well to mention one which has frequently been overlooked; that is, that when the photocell is included in circuit with a high resistance and a constant source of e.m.f. the voltage on the cell will depend on the current flowing in the circuit [26, 27, 28].

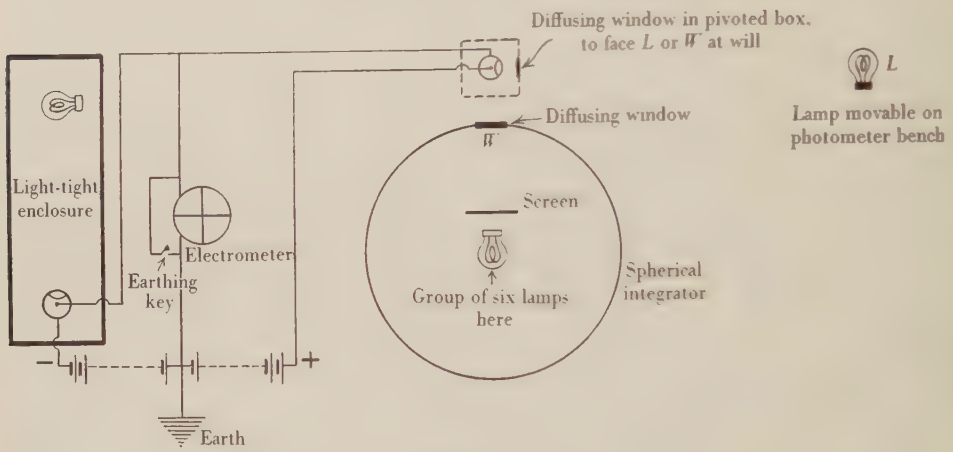

Figure 1.

§ 3. DESCRIPTION OF APPARATUS

For direct measurements of response curves a thermionic bridge photometer was used [29, 24], the photocell, grid resistance and associated valve being enclosed in an earthed screening-box while the rest of the apparatus was unscreened. The photometric system was similar to the sphere and associated photocell in the figure. The sources of light were enclosed in an integrating sphere of diameter 1 m. and consisted of six 100-V., 30-W. vacuum lamps made specially to ensure uniformity of rating, constancy of light output, and good repetition performance. The photocell received light from a small diffusing-window in the integrating sphere. The six lamps were mounted on a common baseboard and separately wired to a junction system outside the integrator. Any lamp or group of lamps could be connected thus to a pair of $\frac{1}{2}$-inch-square copper bus-bars by means of stout copper bridges inserted into mercury cups. A constant voltage was maintained on the bus-bars during a set of observations. Thus it was possible to increase the illumination on the photocell in steps by switching in the lamps one at a time. The observed value of photoelectric current for any group of lamps alight together was then compared

with the figure obtained by adding readings for individual lamps lit separately, and so the deviations from linearity of response could be noted. The object of using lamps of uniform rating was not simply to attain equal steps of illumination (these were not to be expected, if only on account of the different positions of the lamps in the integrator) but to ensure that light of one colour should be used. This precaution was shown to be justified by work on the variation of colour response with light-intensity, an effect which had been pointed out previously [23].

It was in the case of a photometer of the null type that the authors noticed the effect of change in the colour temperature of the source on the illumination response characteristic of the cell. The photocell could be illuminated either with light direct from a lamp on a photometer bench, or by means of the window of an integrating sphere containing a comparison lamp. In ordinary photometry the photocurrents under these two conditions are made equal by adjusting the position of the lamp on the bench. Equality is assured by balancing the current against a constant current from a second independent photocell, an electrometer being used as a null instrument. This photometer is called the *electrometer photometer* hereafter, and is shown diagrammatically in figure 1.

§ 4. PRELIMINARY RESULTS

Tests with the thermionic bridge photometer soon showed that for some photocells, at least, evidence of non-linearity was apparent and that in nearly every case the sensitivity of the cell diminished as the illumination was increased. The deviations from linearity were not generally so great as those quoted by Carruthers and Harrison [30] in their investigation on Talbot's law, but were of the same sign. It thus remained to eliminate the possible sources of error in the apparatus before attempting further precise results.

Another phenomenon was however noticed in using the electrometer photometer (figure 1). In order to investigate whether the diffusing window of the cell-box in this photometer were satisfactory it was decided to utilize the bank of six lamps already available. These were placed in the integrating sphere instead of the usual comparison lamp. They were lit singly and in groups, and the corresponding distances of the bench lamp to give the same photocurrents were observed. If these distances were d_1, d_2, ... d_6, d_{1-2}, d_{1-3}, d_{1-4}, etc., then on the assumption that equality of photocurrent denotes a definite proportionality between sphere illumination and bench illumination, the colour of the sources remaining unchanged throughout, we should have

$$\frac{1}{d_{1\cdot 2}^{\,2}} = \frac{1}{d_1^{\,2}} + \frac{1}{d_2^{\,2}},$$

$$\frac{1}{d_{1-3}^{\,2}} = \frac{1}{d_1^{\,2}} + \frac{1}{d_2^{\,2}} + \frac{1}{d_3^{\,2}} \text{ and so on,}$$

provided that the diffusing window were satisfactory and that its surface might justifiably be taken as the bench origin. It was soon found, however, that the equations were not precisely obeyed and, further, that the diffusing window was not

the cause of the discrepancies, since other experimental conditions changed the magnitude and even the sign of the error. The factor having the greatest influence appeared to be the colour difference between sphere lamps and bench lamp. Accordingly the experiments were repeated with various colour differences, the balance of illumination being maintained for the same lamps run at different voltages by means of stops on the window of the integrator. The various bench distances were thus made very closely the same in every case, so that any influence of poor diffusion of cell-window would be the same for every set of measurements.

In a particular example in which the bench lamp was run at 65 V. the value of $\frac{1}{d_{1\text{-}6}^2} \Big/ \sum_1^6 \frac{1}{d^2}$ was 0·988 when the sphere lamps were at 100 V., and 1·001 when the sphere lamps were at 65 V.

The repetition accuracy of the photometer was at least 0·1 per cent on a single reading, and two readings were taken on each lamp or group of lamps, so that the difference between the two ratios just given is a real one. It can be accounted for only by supposing that the sphere lamps interact with each other differently at different voltages, or that the departure of the photocell characteristic from linearity over the same range of photocurrents is different for light of different colours. The latter explanation proved to be correct, as is shown below. The cell used was the rubidium thin-film vacuum cell again mentioned below.

§ 5. TESTS ON THE THERMIONIC BRIDGE APPARATUS

Tests on the connections of the six lamps showed that each lamp was without influence electrically on the rest, the voltage-drop on the bus-bars with all lamps burning being much too small to cause an appreciable drop in the candle power of the set of lamps, as compared with the sum of individual candle powers. Tests were also made photometrically by covering four lamps with a light-tight box. No change in the candle power of the remaining two was detected on switching the others on and off. Repetition performance of the lamps and photometer together was found to be better than $\frac{1}{4}$ per cent in general.

This allows for changes in sensitivity of the cell, if any, and in several cases possible changes in transmission of coloured glasses with change in temperature. As a check on consistency, in the later work, every set of observations was repeated and in no case was the difference between the two sets sufficient to throw doubt on the validity of the results.

Tests on the thermionic bridge photometer circuits included tests on possible photosensitivity of the valve associated with the photocell and the usual examination for steadiness of zero. In addition the precaution was taken of selecting anode and grid resistances having a low temperature coefficient, so as to minimize drift. The value of grid resistance generally used was 50 megohms, higher values being avoided where possible in order to minimize leakage difficulties. With a sensitive bridge galvanometer and suitably scaled grid potentiometer the overall sensitivity was ample.

The following blank experiments were made to show that the observed non-linearity was really due to the photocell and not to the electrical circuit. (1) With the cell darkened, known potentials were applied to the valve grid circuit in the thermionic bridge by means of a carefully standardized potentiometer. The readings of the photometer were found to be strictly proportional to the applied potentials. (2) In order better to simulate experimental conditions, the photocell and the battery supplying it were replaced by a standard 100-megohm resistance and potentiometer (used to apply, as well as measure, potential). The photometer scale readings were again found to be proportional to the scale readings of the standard potentiometer to within about 0·05 per cent from full scale to one-tenth of full scale. (3) Finally, two photocells were taken, and their illumination response characteristics found by means of the thermionic bridge photometer. They were then connected together in opposition in the photometer so that the apparatus then measured the differential photocurrent. Further, a shutter over one cell was adjusted to make this differential current zero for any one of the bank of six equal lamps, and then fixed. A set of readings on two to six lamps then gave the difference in output between the two cells over this illumination range. It was found to be equal, within the limits of observational error, to the value deduced from the difference between the individually measured characteristics. Since this method is a null method for the detection of non-linearity so far as the thermionic bridge circuit and grid resistance are concerned, it eliminates uncertainty which might arise from poor insulation of the latter and demonstrates the validity of the non-linear characteristics as given by the photometer employed for this investigation.

It should be noted that the effect of any difference in wave-length characteristic between the two cells used in this test was guarded against by selecting accurately colour-matched lamps. The colour-matching was done with the actual two-cell circuit described, so that the errors in matching could not have been responsible for the observed quantities, which were well outside the range of repetition accuracy. Attention may also be drawn here to the error in colour-matching which may ensue from the use of the usual two-cell circuit if the lamps to be matched differ widely in output, and if the two cells have different illumination characteristics.

§ 6. TABLES OF RESULTS

The thermionic bridge circuits, tested as described above, were used to obtain the following sets of observations. The illumination range was in every case only about six to one. Higher ranges could be no doubt obtained by the use of more lamps. The main reason for using only six in the present investigation was the avoidance of undue drift in the photocell and of changes in the transmission through coloured filters due to temperature changes.

It is worthy of note, as Steinke points out[9], that some investigations have covered a wide range of values of illumination by subdividing this range into shorter sections, each of which is related to the next by some independent photometric comparison. In this case the error in the photometric comparison is introduced once

for every section covered and, if large compared with the deviation of the cell from linearity of response, may render the extension of the illumination range valueless.

The tables of results given below are selected from the whole of the work done and are therefore dated to show chronological order. In addition, the tables are divided into three sections, dealing with (i) earlier preliminary measurements with white light only, (ii) later measurements with coloured light, and (iii) one exceptional case. In every case relative sensitivity on an arbitrary scale is tabulated against the number of lamps alight. The mean photocurrent for the six single lamps is indicated also.

The figure for relative sensitivity is obtained by dividing the photocurrent corresponding to a group of lamps by the sum of the currents for the individual lamps lit separately. For a linear cell it would therefore be unity in every case. The cells used were as follows. The cathode material is as stated, to the best of the authors' knowledge.

Osram KMV6, No. 12255. Potassium on copper thin film, central plate cathode. Excess potassium in base of cell.

Osram infra-red cell. Caesium on silver thin film. Central plate cathode.

Gramophone Co. cell. Rubidium on silver thin film. Cathode on bulb, central anode.

Osram KMV6, No. 21076. Potassium on silver thin film, central plate cathode. No excess potassium.

Osram KV6, No. 21460. Bulk potassium, central plate cathode.

Philips Type 3512. Cathode on bulb, metal not stated.

Philips Type 3515. Cathode on bulb, metal not stated.

*Osram KMV6, No. R*0018. Potassium on silver thin film, central plate cathode.

(i) *White light.* Results for Osram Vacuum cell of type KMV6, No. 12255 (old type). The presence of a quantity of potassium in bulk in this cell caused instability of both wave-length and illumination characteristics with lapse of time.

Set 1. Date 17. v. 1932.

Number of lamps	Relative sensitivity		
	Cell volts 58	Cell volts 136	Cell volts 294
1	Unity (0·01 μA.)	Unity (0·01 μA.)	Unity (0·01 μA.)
2	0·994	0·995	0·999
3	0·989	0·992	0·995
4	0·986	0·990	0·992
5	0·984	0·988	0·997
6	0·983	0·985	0·997

Set 2. Date 14. xi. 1932.

Number of lamps	Relative sensitivity		
	Cell volts 12	Cell volts 22	Cell volts 62
1	Unity (0·007 μA.)	Unity (0·007 μA.)	Unity (0·007 μA.)
2	0·995	0·994	0·994
3	0·992	0·991	0·990
4	0·991	0·989	0·987
5	0·990	0·987	0·985
6	0·989	0·986	0·984

Results for Osram infra-red cell.

Set 3. Date 24. v. 1932.

Number of lamps	Relative sensitivity cell volts 58
1	Unity (0·0105 μA.)
2	0·998
3	0·997
4	0·996
5	0·995
6	0·997

Set 4. Date 14. xi. 1932.

Relative sensitivity cell volts 22
Unity (0·007 μA.)
1·000
0·999
0·998
0·997
0·997

Results for Gramophone Co. cell.

Set 5. Date 17. v. 1932.

Number of lamps	Relative sensitivity cell volts 58
1	Unity (0·0096 μA.)
2	0·992
3	0·988
4	0·986
5	0·984
6	0·983

Set 6. Date 14. xi. 1932.

Relative sensitivity cell volts 22
Unity (0·007 μA.)
0·994
0·992
0·992
0·991
0·991

In addition to these results measurements were made on the KMV6, No. 12255 cell with and without the filter used for colour correction in ordinary photometry (a green liquid filter). There was some indication that the results were slightly different according as the filter was used or not.

(ii) *Coloured light.* The following results were obtained during July, 1933, on seven cells of different types, six of which were supplied as suitable for photometric purposes. Three colour-filters were used in conjunction with them to select different spectral ranges. The same range of photocurrents was used for different tests on the same cell, in order that the electrical conditions in the cell might be comparable in the different cases. This necessitated adjustment of the illumination in accordance with the relative colour sensitivity of the cell, since light of different colours was used. The cell-voltages were those recommended by the makers with the exception of the rubidium cell, which was tested at two voltages to determine

whether its spherical shape and large size were likely to lead to discrepancies due to non-attainment of saturation; and of the Philips cell of type 3515. This cell was probably gas-filled, as the results seem to indicate, and a voltage of 26 was arbitrarily assigned to it.

Furthermore, on the advice of manufacturers, the photocurrents were of smaller value than previously, so as to ensure definitely that the cells were not overloaded.

The coloured filters used consisted of Chance glasses. The blue cobalt glass had a transmission of 21 per cent at 4200 Å. falling to 1½ per cent at 4800 Å. The green glass transmitted 34 per cent at 5400 Å., the value falling to 4 per cent at 6000 Å. and 4800 Å. The red selenium glass transmitted about 86 per cent from the infra-red to 6700 Å. and cut off sharply at a wave-length slightly shorter. The lamps were not operated at the same voltage in the case of each filter, but the voltage changes were always such as to emphasize the colour-changes imposed by the filters.

Set 7. Osram KMV6 cell No. 12255 at 22 V. Photocurrent for one lamp, 0·00101 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·999	0·989	0·998
3	0·995	0·981	0·997
4	0·994	0·976	0·997
5	0·993	0·972	0·995
6	0·991	0·966	0·994

Set 8. Osram KMV6 cell No. 21076 at 22 V. Photocurrent for one lamp, 0·0019μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	1·000	1·000	0·999
3	1·000	1·000	0·999
4	1·000	1·000	0·999
5	0·999	0·999	0·999
6	0·999	0·999	0·999

Set 9. Osram KV6 cell No. 21460 at 22 V. Photocurrent for one lamp, 0·00051 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Readings unobtainable. Sensitivity too small
2	0·998	0·999	
3	0·997	0·999	
4	0·997	0·998	
5	0·997	0·998	
6	0·997	0·997	

Set 10. Osram infra-red cell at 22 V. Photocurrent for one lamp, 0·00085 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·998	1·000	1·000
3	0·994	1·000	1·001
4	0·992	0·999	1·001
5	0·990	0·999	1·001
6	0·988	0·999	1·001

Set 11. Gramophone Co. cell at 22 V. Photocurrent for one lamp, 0·0014 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·951	0·990	0·999
3	0·926	0·980	0·999
4	0·909	0·974	0·999
5	0·899	0·969	0·997
6	0·891	0·965	0·996

Set 12. Gramophone Co. cell at 138 V. Photocurrent for one lamp, 0·0015 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·966	0·989	0·998
3	0·940	0·985	0·998
4	0·920	0·980	0·999
5	0·901	0·977	0·999
6	0·893	0·974	0·999

Set 13. Philips cell, type 3512, at 22 V. Photocurrent for one lamp, 0·0012 μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·938	0·985	1·000
3	0·904	0·973	0·996
4	0·880	0·967	0·994
5	0·861	0·960	0·993
6	0·848	0·957	0·990

Set 14. Philips cell, type 3515, at 26 V. Photocurrent for one lamp, 0·0014μA.

Number of lamps	Relative sensitivity with		
	Blue filter	Green filter	Red filter
1	Unity	Unity	Unity
2	0·927	1·001	1·008
3	0·894	1·001	1·014
4	0·873	0·999	1·016
5	0·860	0·998	1·017
6	0·850	0·997	1·018

The apparent rise in sensitivity for red light in the case of the Philips 3515 cell may be due to the gas-filling or to a rise in temperature affecting thermionic emission. The latter cause is improbable, however, since a repeat reading on one lamp immediately after one on the group of six lamps confirmed the previous reading on the same lamp. The presence of a small residuum of gas may account for the upward bend in the results of set 10.

(iii) Set 15. Date 11. x. 1933. Osram KMV6 cell No. R0018 at 22 V. Photocurrent for one lamp, 0·002μA. Tested with 100-volt lamps running at 53 V. No colour-filter.

Number of lamps	Relative sensitivity
1	Unity
2	1·000
3	0·998
4	0·997
5	0·995
6	0·995

In the case of set 15, the sensitivity corresponding to exposure to light from a group of lamps persisted for some little time after extinguishing the lamps, returning gradually to the original state for one lamp. No such prolonged change was observed in any other case, though tests were always made to detect it if possible. Set 15, therefore, is not explained by the theory given below.

§ 7. SUMMARY OF RESULTS

The above results show that in the employment of a photocell for precise photometry no assumption as to its linearity should be made. On the other hand they show that a carefully made modern cell may prove on test to be linear. Again, a cell may be linear for light of one colour and markedly non-linear for light of another colour. The bearing of this on photoelectric spectrophotometry is obvious. It implies too that curves of spectral sensitivity of a cell will depend on the illumination-values used for the calibration[23] and is in fact a sort of Purkynè effect, as was pointed out to the authors by Dr J. W. T. Walsh. The bearing of the effect on heterochromatic photometry in which photoelectric cells are used with a null method is perhaps not at first obvious, though it is exactly analogous to the similar case in visual photometry.

Suppose that $\phi_1(E_1)$ represents the photocurrent i for illumination E_1 of colour temperature T_1 and that $\phi_2(E_2)$ represents that for illumination E_2 of colour temperature T_2; then in a null method, we adjust E_1 and E_2 to give equal photocurrents, say i'. We then have

E_1
T_1, E_2
T_2

$$i' = \phi_1(E_1') = \phi_2(E_2').$$

If we make another setting at another value i'' of photocurrent (e.g. by bringing both sources closer to the cell) we have (say)

$$i'' = \phi_1(E_1'') = \phi_2(E_2''),$$

but since ϕ_1 and ϕ_2 are not the same function, we cannot deduce that

$$E_1'/E_2' = E_1''/E_2'',$$

which is the criterion for absence of the Purkyně effect.

The results described above and obtained with the electrometer photometer are thus explained by the fact that the rubidium cell used had distinctly different illumination characteristics for different regions of the spectrum (see set 11).

It is also noticeable that in almost every case the cell's sensitivity, if not constant, falls with increasing illumination. The few exceptions include one case of a cell (Philips 3515) which possibly was gas-filled and which in any case was of a type generally considered unsuitable on other counts for photometric purposes.

Finally it is clear that few cells have a response curve sufficiently near to the linear form to justify their employment, without the use of correction factors, in high-precision photometry to an accuracy of $\pm 0{\cdot}1$ per cent, except where the light-sources under comparison have very similar spectral distributions, and the cell is used at constant illumination. Where a cell is used for colorimetric or spectrophotometric purposes its illumination characteristic must be examined with light of the colour or colours for which it is to be used.

§ 8. POSSIBLE CAUSES OF NON-LINEARITY

In discussing the causes of non-linearity we may dismiss any effect of residual gas in the cell and confine attention to cells of modern manufacture in which the vacuum is hard enough to render such effects negligible. Earlier "vacuum" cells are less reliable in this respect.

Space-charge effects. Failure to attain saturation, due to space-charge effects, might be advanced as a possible cause. This is, however, discounted by the fact that the available evidence seems to show that the cell-sensitivity falls off less rapidly the higher the illumination, whereas space-charge effects become more serious. Again, results for blue and red light, as given above, were taken for the same range of photocurrent, and space-charge effects would presumably be the same in the two cases. The corresponding cell-characteristics are, however, often markedly different. On the other hand in some cases increase of cell-voltage appeared to improve the cell's performance, as would be expected if space-charge effects were in question.

Thermal effects. A second possible cause, namely drift due to heating of the cell by the incident radiation, is unlikely to have been operative during the recorded observations because, with the exception of the cell used for set 15, none of the cells tested showed evidence of such drift when exposed to the illumination from the six-lamp group for a period of time. Further, repetition of a single-lamp reading immediately after the six-lamp exposure showed no evidence of a change in cell-sensitivity due to the previous exposure to a higher illumination. It may also be noted that each complete set of observations took only ten minutes. Tests with a cupric chloride cell, absorbing the infra-red, also gave negative results on this point.

Effect of thin films of photoelectric material. It seems probable that the main cause of non-linearity is still the presence of films of photoelectric material in poor electric connection with the main cathode[32]. These films may be on the glass walls, thus constituting a high resistance in themselves, or may perhaps be upon the surface of the cathode, but separated from it by a layer of a poorly conducting material. The latter formation is suggested at once by the method of preparation of many thin-film cells, in which the cathode plate is oxidized before deposition of the alkali metal. An oxide layer of the ideal thickness cannot probably be regarded as a semi-conductor, since it is of molecular dimensions (or of that order), but the presence of unavoidable thicker portions may provide ground for the supposition.

The photocurrent may then be supposed to come partly from a portion of the cathode in direct metallic connection with the cell terminal and giving a linear response, and partly from other portions or films in poor contact with the cell terminal. The potential-drop in the film due to the passage of the photocurrent will cause part of the film to reach such a voltage as to prevent electrons from escaping. The area of this part will vary with the photocurrent in the film, and so non-linear response will ensue. The overall characteristic of the cell will then be the sum of a linear characteristic and a non-linear one, due to different photosensitive areas in the cell. The change in characteristic with change in colour of the incident light then naturally follows from the fact that these areas have different spectral sensitivities due to their different structures and work functions.

An endeavour is made in the next section to develop the isolated-film hypothesis to account quantitatively for the observed results. Before proceeding to the theoretical considerations, however, we may note that the hypothesis gives a clear picture of the reason why the results of Carruthers and Harrison[30] show that non-linear cells obey Talbot's law. For if the illumination is flickered sufficiently rapidly, the current from the high resistance will take up an equilibrium value (probably not large enough to be detected during the period of occultation on Carruthers's oscillographs[34]), owing to lag in leakage of charges developed by individual flashes of light. This is equivalent to the sort of "fatigue" suggested by Campbell[33], though he suggested temperature change as the cause. The cell will then of course behave in the same way as it would for a steady illumination of the same mean value. This will be so independently of the type of steady (illumination, response) function that may arise from the isolated-film hypothesis. Thus the equation given by

Stiles[31] for the form of the illumination characteristic of cells obeying Talbot's law may not be unique, and in fact is not borne out by the observations already tabulated.

§ 9. THEORETICAL TREATMENT

Let AB represent a conducting photosensitive film of resistance R per unit length and connected at A with the cell cathode at potential P equal to P_0. Suppose that at some point X in the film the potential P, equal to P_1, is such that between A and X photoemission takes place while between X and B it is stopped. Such a precise line of demarcation will not in practice exist, but is here assumed in order to render a simple mathematical treatment possible. Consider a strip of the film, dl in length, distant l from X. Let the photoemission from it be di and the currents flowing in and out of the strip, as shown, be $i + di$ and i. Let the potential drop across the strip be dP. Let the incident illumination be E.

R

P, P_0

P_1

l

i

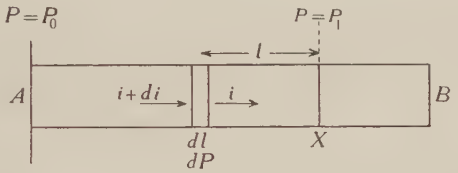

Figure 2.

Then we have

$$i = kEl \text{ where } k \text{ is a constant} \quad \ldots\ldots(1), \qquad k$$

$$dP = iRdl$$

$$= kERldl \text{ by (1)}$$

and $$P - P_1 = \tfrac{1}{2}kERl^2 \text{ by integration.}$$

Thus the total length l_0 of the photoelectrically active part of the strip is given by l_0

$$P_0 - P_1 = \tfrac{1}{2}kERl_0^2,$$

and $$l_0 = \sqrt{\left(\frac{2}{k}\cdot\frac{P_0 - P_1}{ER}\right)} \quad \ldots\ldots(2).$$

Now the total current from the strip $l_0 = kEl_0$

$$= \sqrt{\frac{2kE(P_0 - P_1)}{R}}.$$

Thus the illumination characteristic for the strip is of the form

$$\text{current} \propto \sqrt{\text{illumination}}$$

or $$\text{sensitivity} = \frac{\text{current}}{\text{illumination}} \propto \frac{1}{\sqrt{\text{illumination}}}.$$

If now to complete the picture of the photocell we add a portion of cathode surface obeying a linear law, we obtain for S, the sensitivity or current ÷ illumination, an expression of the form S

$$S = K_1 + K_2/\sqrt{E} \quad \ldots\ldots(3),$$

where K_1 and K_2 are constants, provided only that there remains some of the K_1, K_2

strip AB which is inactive. Whether this is so or not will depend on the value assigned to P_0, the value of R or the value of E.

The value of R cannot be controlled externally, but if P_0 or E is adjusted so that l_0 as given by equation (2) is greater than AB, the whole of the strip will be active and the cell will be linear. Again, for large values of E, the term $K_2\sqrt{E}$ in equation (3) will be small compared with K_1 and the characteristic of the cell will become more nearly linear, corresponding to a limiting sensitivity S equal to K_1. The (sensitivity, illumination) curve will then consist of a short horizontal portion for small values of E, leading into a curve which falls somewhat abruptly at first, but which for higher values of E tends asymptotically toward a constant ordinate K_1. This general shape, including the horizontal portion near $E = 0$, has been confirmed by work on one cell only over an extended range of illumination, but it is not desired to present incomplete and in some respects inconclusive evidence at the present juncture. Establishment of the illumination scale to the desired accuracy is difficult over the required range. Moreover the flat part of the curve did not in this case extend beyond the point corresponding to a photocurrent of about $0{\cdot}0005\,\mu$A. Measurements of both current and illumination below this point require considerable care.

In practice the assumptions as to the shape of the strip AB, and the existence of a line at X dividing the active from inactive portions of the strip, will not be fulfilled. But it is interesting to inquire whether the equation $S = K_1 - K_2\sqrt{E}$ fits the observed values given earlier in this paper.

In the case of set 13 (blue filter) we have

S observed:	1·000	0·938	0·904	0·880	0·861	0·848
S calculated; $K_1 = 0{\cdot}726$, $K_2 = 0{\cdot}30$:	1·026	0·938	0·899	0·876	0·860	0·849

These calculated values show that if a good fit with observed values is obtained over the lower part of the curve, the upper observed value is low, as might be expected if the curve rounds off into a flat portion near $E = 0$. On the other hand a moderately good fit over the *whole* curve may be obtained with other values of K_1 and K_2. The general shape indicated by the theoretical treatment is, however, borne out.

If we take set 11 (blue filter) even better agreement can be obtained, thus:

S observed:	1·000	0·951	0·926	0·909	0·899	0·891
S calculated; $K_1 = 0{\cdot}810$, $K_2 = 0{\cdot}20$:	1·010	0·951	0·926	0·910	0·899	0·892

The theory put forward above thus seems to fit the observed facts, but it is not claimed that the form of equation given is unique as regards agreement with practical results. Finally it must be emphasized that if the difficulties inherent in the use of a wider illumination-range can be satisfactorily overcome, extension of the results to such a range is most desirable.

§ 10. CONCLUSIONS

Measurements on a group of photocells of modern construction show that while cells are obtainable which give almost strict proportionality between photocurrent and illumination, it is unsafe to employ a cell for precise photometry without a linearity test. Moreover, since deviations from linearity may differ according to the colour of the incident light, calibration should, if necessary, be carried out at several different wave-lengths.

In almost every case the cell sensitivity falls with increasing illumination and may be expressed by an equation of the form $S = K_1 + K_2/\sqrt{E}$, which is developed upon the assumption that badly conducting photosensitive films present in the cell are responsible for non-linearity. This assumption is borne out also by the observed fact that deviations from linearity are greater the shorter the wave-length of light used, for the metallic film will probably have bulk-metal properties with maximum photosensitivity at shorter wave-lengths, while the main cathode contributing a linear quota of photocurrent will be of the oxide-layer type with maximum sensitivity toward longer wave-lengths.

The hypothesis of a poorly conducting film also explains the recorded fact that non-linear photocells obey Talbot's law, irrespective of whether the equation developed above is correct or not.

The resemblance of the non-linear properties of photocells to the Purkyně effect in the eye is mentioned, and the bearing of these upon heterochromatic photometry by null methods is pointed out.

§ 11. ACKNOWLEDGMENTS

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DISCUSSION

Dr J. W. T. Walsh. This paper settles at last the long-standing discussion as to whether the response of photoelectric cells can be taken as linear or not. It is clear that the assumption cannot be made and that cells must be tested for this property. The Research Laboratories of the General Electric Company showed at the Society's recent exhibition the apparatus used at Wembley for making this test and I hope Mr Winch will tell us more about their experience in this matter.

I should like to ask the authors whether they have found the sensitivity curves of cells to remain constant over extended periods either of use or of rest. Further, do they consider that, for work of the highest accuracy, no reliance should be placed on the linearity of the cells, if all measurements are made with the same value of the illumination? I hope they will continue their work and that, in particular, they will investigate more fully the form of the curve and especially the initial portion of it.

Mr G. T. Winch referred to the substantial departure from linearity exhibited by the cell used in set 7 of the observations, as compared with the high degree of linearity shown in sets 8 and 9. This was possibly due to the fact that some of the excess potassium which is present in the older type of cell may have been deposited on the cathode as the result of very slight warming such as can occur through handling. The cells used in sets 8 and 9, being of much more recent manufacture, are not subject to this trouble. The speaker had tested several random-selected KMV6 cells with a developed technique and found that in three or four of them with light from tungsten vacuum lamps the departures from linearity did not exceed 5 parts in 10,000. It would be interesting to see the authors' individual measure-

ments; a comparison of the readings from set to set would indicate whether the claim for sensitivity was fully justified. It was surprising that sensitivity was shown always to decrease with increasing luminous intensity; the speaker had found small apparent fluctuations of sensitivity which he was inclined to think were as likely to be errors of measurement as changes in cell sensitivity, but he had not observed consistent unidirectional change with changing sensitivity.

Mr J. Guild. The discrepancies discussed in the paper, are, of course, merely an example of what is always found when any physical phenomenon is utilized in apparatus intended for measurements of high accuracy: the apparatus never behaves in accordance with the simple laws which we are led to expect from the text-books or the academic investigations. In most of the earlier work on photoelectricity the emphasis has lain on the theoretical significance of the strict proportionality between intensity of illumination of any constant spectral energy-distribution and photoelectric emission as such. While we may have no present reason to doubt that this strict proportionality exists, it is necessary to bear in mind that our evidence for it is only its theoretical probability and not direct experimental confirmation, because any measurements we make must necessarily be made on the currents delivered not by a theoretically simple photosensitive surface but by a photoelectric cell complete with contact resistances, internal resistance and so on, which have long been suspected of producing disturbing effects of the kind the authors have here demonstrated. I think there is no doubt that the kind of schematic explanation of the departure from linearity given in the paper, though not necessarily the only possible one (as the authors admit), is substantially correct in its main features. I think also that the same underlying causes are responsible for the fatigue exhibited in very different degrees by different cells. I have not, myself, much direct acquaintance with the behaviour of vacuum cells, but I have made considerable use lately of various types of rectifier or photovoltaic cell. Some types of rectifier cell exhibit enormous fatigue effects, the current for a given illumination falling asymptotically to about a tenth of its initial value after 30 minutes' exposure or more, while other types show negligible or quite minute fatigue. While I have not examined these cells with exactly the same objects in view as the authors of the present paper, there seems to be a definite correlation between the magnitude of the fatigue effects and the departures from linearity of the (current, illumination) relation. This must not, of course, be confused with the departures from linearity due to external resistance in the galvanometer circuit, which all such cells exhibit and which is due to a different and well-known cause.

Without reference to underlying causes it is obvious that a cell which exhibits fatigue must also give a non-linear response, and, further that the departures from linearity will depend to some extent on the time sequence of the observations. Where the effect is marked, it should be appreciably different on the ascending and descending loops of a cyclic series of intensity-variations. I should like to know if the authors have found any such differences in the cells they have examined.

In reply to a question (asked later) by Dr Stiles, I do not know how accurately rectifier cells which exhibit marked fatigue obey Talbot's law, as I have not used sector discs with them except for quite rough purposes. I should say that they certainly obey it to within one or two per cent, but I have made no tests of greater precision.

As regards the practical aspects of the authors' results, they confirm the view that I and many others have always held, that when the highest quantitative accuracy is sought one should never rely on the assumed proportionality of response to stimulus, even with cells in which tests have failed to reveal a definite discrepancy, but should always adopt an experimental arrangement in which all the cell factors are eliminated. In work with *gas-filled* cells it is, in fact, desirable to ensure that the stimulation of the cell never alters during the whole of the processes involved in comparing two illuminations. If, for example, such a cell is exposed to one illumination, and is then left dark while a second illumination is brought to bear on it, it does not, in practice, follow that the second illumination is equal to the first when adjusted to bring the response of the cell to the same value. We have found it necessary, for substitution methods of this kind, to arrange matters so that a constant illumination falls on the cell during the complete process of substitution. The causes of difficulties of this kind may reside entirely in the gas, but they may possibly be in some part due to fatigue effects in the film, and I should be interested to hear if the authors have found any such effect with vacuum cells.

Dr W. S. Stiles. The authors' explanation of the cause of non-linearity is of particular interest. The agreement of their experimental results with a derived law $S = K_1 + K_2/\sqrt{E}$ raises the question whether somewhat different assumptions as to the shape and other properties of the thin films would lead to the same result. The authors point out that their hypothesis would fit in with the obedience to Talbot's law of non-linear cells, because it gives a mechanism for a short-period fatigue in the cell's response. Some time ago I discussed the connection between non-linearity, fatigue and obedience to Talbot's law in photoelectric cells, and derived the formula $S = K_1 - K_2E$ for the sensitivity[31] assuming fatigue and recovery to obey a simple exponential law. The present accurate measurements show this result to be invalid, but there must still be a very close connection between the form of the fatigue curve and the shape of the response characteristic. A study of the fatigue curve, possibly by the method mentioned in my paper[31], might lead to an estimate of the order of magnitude of the resistance and capacity of the thin film which are involved in the authors' theory.

Mr Guild mentioned the correlation he had observed between non-linearity and fatigue effects in various photovoltaic cells. It would be interesting to know whether the cells in question obey Talbot's law.

Mr G. P. Barnard. I hope that the authors will regard equation (3) as an approach to a more general mathematical treatment. I do not feel that the theory

given in this paper for these cells can be directly applied, as has been suggested by Mr Guild, to the observed deviations from linearity in response of the Sperrschicht type of cell. An investigation at the National Physical Laboratory on the Sperrschicht cell is in progress; the results show definitely that other factors determine the form of the illumination-response characteristics. It is intended to publish shortly a paper on this work.

Authors' reply. Dr Walsh has raised a question of perhaps greater importance than the simple consideration of linearity of response in asking whether the response curve remains the same for a particular cell over a period of time. For cells of the vacuum type containing no excess alkali-metal the answer is, fortunately, yes, so far as our experience is concerned. This is also a reply to Mr Winch's question as to the repetition accuracy of the quoted results. Several cells mentioned in the present paper showed consistent behaviour over a period of about a year. As regards the employment of photocells for photometry to an accuracy of $\pm$ 0·1 per cent, we consider that no assumptions as to linearity of response should be made. As Dr Walsh suggests, measurements should be carried out at constant illumination. Even then, colour differences are to be avoided where possible in order to exclude the type of Purkyně effect mentioned in the paper, which, however, generally involves errors of a smaller order than those arising from the employment of the cell for non-constant illumination. It is agreed that further work on the shape of the response curves is most desirable.

Mr Winch contrasts the results of set 7 with those of sets 8 and 9. We agree with his suggestion that the poorer performance in the former case may be due to the presence of free potassium which encourages the formation of stray films of metal. Mr Winch also stresses the systematic downward trend of the sensitivity curves we have measured, stating that his own observations had in general shown fluctuations in sensitivity above and below some selected normal value, as the illumination was increased. We would suggest that on prima facie physical considerations the former behaviour would seem the more probable. Moreover, the appearance of this systematic behaviour in our results is a strong argument in support of the claim that purely fortuitous error has been reduced to a very low order in the experimental work.

We cannot at the moment give a reply to Dr Stiles's question as to what laws of response would arise from the mathematical investigation of thin films of different shapes. In the case of the cells used by us the shape most likely to arise would be roughly rectangular. As Dr Stiles points out, there must be a close connection between the static response characteristic of a cell and its behaviour under flickered illumination. In fact the general argument advanced in his paper[31] must clearly hold even if the present results indicate that some of his simplifications are not justifiable. Investigation of the average output of a cell at various frequencies of interruption of the light should certainly lead to further knowledge

concerning the resistance and capacity of the stray metal films. Alternatively we suggest observing by means of an oscillograph the form of the discharge from these films immediately upon the cutting off of a steady illumination.

Such cells of the rectifier type as have come to our notice have all shown obedience to Talbot's law for high frequencies of interruption. Mr Guild carries over the relation of fatigue and non-linearity to the case of the rectifier cell. While no doubt similar effects of metallic films occur here also, other factors certainly have an influence on the response characteristic of the cell, since in general it appears that the characteristic for a linear rectifier cell is a happy combination of a non-linear voltage curve with a non-linear resistance curve. In these respects we are indebted to Mr Barnard for information and look forward to seeing a fuller account of his recent work on this interesting type of photocell in the near future.

Reverting to the question of fatigue in general, it is convenient to distinguish between a short-period and a long-period fatigue without prejudice to the possibility of their being eventually ascribed to a common cause. Such fatigue as we have discussed already has been assumed to be of a period short compared with the time taken by the observations, though long compared with a single cycle of illumination in investigations on Talbot's law, for otherwise the law would not hold precisely[31]. There is sometimes observed, however, a long-period effect operative over periods upwards of several minutes. For instance the Osram cell No. R0018 on further investigation showed a drop in sensitivity of over 1 per cent in 10 minutes under steady illumination, and a recovery in the dark at a corresponding rate, the curves being roughly of an exponential type. This was certainly not due to the cause which first suggested itself, namely temperature rise of the cathode, since the incidence of a beam of infra-red light (to which the cell was insensitive photoelectrically) was without effect. An alternative cause put forward by J. H. de Boer* seems more probable and has an interesting bearing on the suggestions made in the present paper.

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THE PREPARATION OF HEAVY HYDROGEN*

By P. HARTECK, Ph.D.

Communicated by Prof. Lord Rutherford, December 7, 1933. Read December 15, 1933.

ABSTRACT. The method of working for the preparation of heavy hydrogen by means of the electrolysis of an alkaline solution with nickel electrodes is described in detail.

G. N. Lewis and R. T. Macdonald† have described the preparation of heavy hydrogen by means of the electrolysis of an alkaline solution with nickel electrodes. They do not give details of their experimental methods. There are, however, some precautions which must be observed if the enrichment of the heavy hydrogen (H^2) is to proceed satisfactorily. This paper gives a description of a convenient method of working. For working this out it was a great advantage that slightly enriched samples of water could be examined in the apparatus of Oliphant and Lord Rutherford‡ in which lithium is bombarded with hydrogen ions. The long range α-particles which are observed are due to collisions involving H^2 according to the equation

$$H^2 + Li^6 = 2\alpha \text{ (range 13 cm.)}.$$

The amount of heavy hydrogen in ordinary water gives a measurable number of disintegration particles, and affords a rapid method of estimating the amount of H^2 at weak concentrations.

About 60 samples of enriched hydrogen, which had been prepared by electrolysis of normal water under varying experimental conditions, were tested in this way. When the method had been developed the H^2 concentration was determined by density-measurements. For the preparation of the highly concentrated samples of heavy water I started with water supplied by the Ohio Chemical Co. containing 0·33 per cent and 0·48 per cent of H^2.

At the cost of power to the Cavendish Laboratory (1½*d.* per kilowatt hour) it was uneconomical to make the initial stages of the electrolysis on the spot, and for that reason the enriched water was obtained for these experiments by Dr Cockcroft during his recent visit to the United States of America.

The electrolytic cells were made of nickel supplied by the Mond Nickel Co. The cells were soldered carefully on the outside so that practically no solder came in contact with the electrolyte. All metal parts which came into contact with the liquid or the condensed water of the cell were of nickel. Figure 1 shows a cell of capacity 5 litres. The cell is placed in a water bath and stands on legs which permit

* H. C. Urey, F. G. Brickwedde and G. M. Murphy, *Phys. Rev.* **39**, 164 (1932). E. R. Washburn and H. C. Urey, *Proc. Nat. Acad.* **18**, 496 (1932).

† G. N. Lewis and R. T. Macdonald, *J. Chem. Phys.* **1**, 341 (1933).

‡ M. L. E. Oliphant and Lord Rutherford, *Proc. R.S.* A **141**, 259 (1933). M. L. E. Oliphant, B. B. Kinsey and Lord Rutherford, *Proc. R.S.* A **141**, 722 (1933).

the circulation of water below the cell. Inside the cell there is a cooling-spiral of glass, not shown in the sketch. The case of the cell is the cathode, and the anode is a perforated sheet of nickel which can be adjusted to be exactly parallel to the bottom of the cell and a few millimetres above it.

The electrolysis is begun with $\frac{1}{2}$ per cent NaOH. With a current of 150 to 200 A.* and cooling-water at approximately 10° C. the temperature of the electrolyte does not rise above 25° C. In spite of the low concentration of NaOH the nickel of the anode does not go into solution because of the high current-density. The surface of the nickel is carefully cleaned before the electrolysis begins. Grease and organic compounds in general must be carefully avoided. The wet surface is scrubbed with glass paper and steel wool and the electrolysis is immediately begun. This

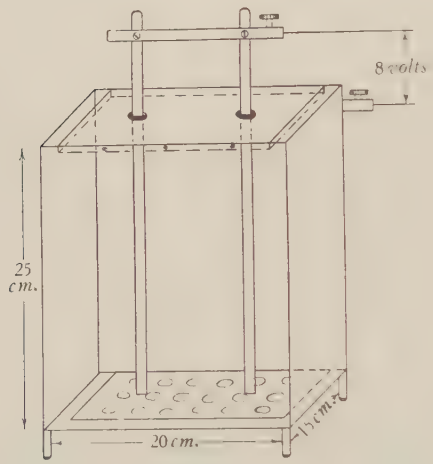

Figure 1.

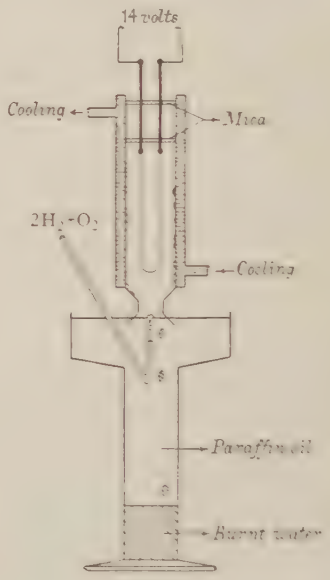

Figure 2.

method has been found to be more convenient than the addition during the electrolysis of a complex nickel salt which produces a new surface layer of nickel. If allowed to stand exposed to the air a nickel surface becomes coated with a layer of oxide which seems to lower the efficiency of the electrolysis. After the electrolysis of one or two lots of solution the surface is again cleaned. The sticks of NaOH are washed with distilled water before the solution is made up.

More than twenty-five electrolyses have been carried out in this manner with four different cells and without any case of failure. Since the initial concentration of alkali is so small, it is possible to electrolyse to $\frac{1}{20}$ or $\frac{1}{30}$ of the initial volume. The solution is then neutralized with CO_2 and the water is distilled off, as described by Lewis and Macdonald.

* Corresponding to a current-density of from 0·6 to 0·8 A./cm².

Let α and α' be the relative rates of removal by electrolysis for H^1 and H^2. Then the amounts of residual H^1 and H^2 at time t are given by the equations α, α' t

$$H_t^1 = H_0^1 e^{-\alpha t} \quad \text{and} \quad H_t^2 = H_0^2 e^{-\alpha' t}.$$

Lewis and Macdonald give a value $\alpha/\alpha' = 5$. In my last series of experiments α/α' was never less than 5·5 and was usually in the neighbourhood of 6·5. In calculating these values the effects of evaporation and losses during distillation were neglected. If corrections are made for these factors α/α' becomes slightly greater. No value of α/α' greater than 7 has been observed. The efficiency of the electrolysis seems to be independent of the concentration of alkali over a wide range. No definite increase of the efficiency was observed with increasing current-density provided this was greater than 0·6 A./cm.^{2} The dependence of the electrolysis on temperature was not investigated since it was always necessary to work at a low temperature to keep down evaporation.

If the electrodes are really clean the efficiency has its full value from the beginning.

Table. Efficiency of separation

No.	V_0 (cm.^{3})	V (cm.^{3})	$\frac{V_0}{V}$	H_0^2 (per cent)	H^2 (per cent)	$\frac{\alpha}{\alpha'}$
1	3000	300	10	0·33	2·3	6·7
4	3000	375	8	0·33	2·0	6·5
6	3000	100	30	0·33	5·3	6
10	110	6	18	12·5	90·0	5
15	40	1	40	12·2	100	—
23	2500	22·5	110	0·48	27	6
26	120	6	20	12	91·5	5·5

As soon as the concentration of H^2 had the value of about 12 per cent, the mixture of oxygen and hydrogen coming off from the cell was burnt to water to avoid loss.

The following method of working is efficient and safe. The gas mixture bubbles through paraffin oil in the bottom of a cylinder containing a white-hot platinum wire, figure 2. The top of the cylinder is made roughly air-tight with thin mica sheets. The gas mixture combines on the surface of the platinum wire. The water which is formed condenses on the water-cooled cylinder-wall and collects under the oil. Before this recovered water is used for further electrolysis it must be filtered to get rid of paraffin oil and distilled with 1 per cent $KMnO_4$.

The exact formula for the change of concentration of H^1 and H^2 at any value of the concentration of H^1 and H^2 is

$$\left(\frac{C_{H^1}}{C_{0H^1}}\right)^{\frac{\sigma}{1-\sigma}} \cdot \left(\frac{C_{0H^2}}{C_{H^2}}\right)^{\frac{1}{1-\sigma}} = \frac{V}{V_0},$$

σ where $\sigma = \frac{\alpha'}{\alpha}$ and is < 1; C_{0H^1}, C_{0H^2} and V_0 are the initial concentrations and volume, C_{H^1}, C_{H^2} and V are the corresponding quantities after the electrolysis. The equation is identical with that of Rayleigh deduced in the theory of fractional distillation. In figure 3 the ordinates are the logarithms of H^2 per cent and the abscissae logarithms of the ratio V_0/V. The curves represent the changing concentration of H^2 and H^1 as the volume diminishes for the two values $\frac{1}{5}$ and $\frac{1}{6}$ of σ,

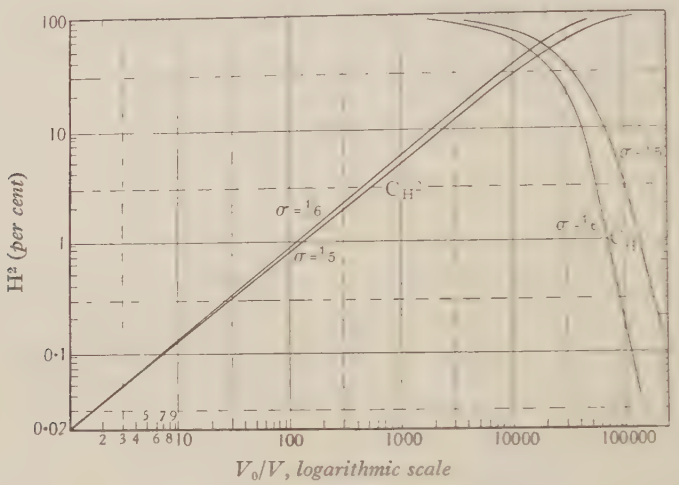

Figure 3.

corresponding to values 5 and 6 of α' α. In this diagram it is assumed that α' α remains constant over the whole range of concentration. This is certainly true as a first approximation—compare experiments 4 and 10 in the table.

In the course of these experiments we have obtained enriched samples of heavy water containing from 30 per cent to 98 per cent of heavy hydrogen, corresponding in all to about 25 cm³ of pure heavy water. Experience has shown that the 30-per-cent samples form a suitable source for obtaining supplies of heavy-hydrogen ions for transmutation experiments.

I am much indebted to Prof. Lord Rutherford and Dr Oliphant for their kind permission to test the concentration of H^2 with their disintegration apparatus, and to the Rockefeller Foundation for a grant.

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A HIGH-VACUUM LEAK DEVICE

By G. A. WHIPPLE, B.A.

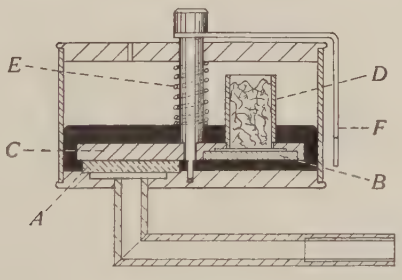
Communicated by Robert S. Whipple, December 13, 1933. *Read in title, January* 19, 1934.

ABSTRACT. The high-vacuum leak device described was designed for the control of pressure in the discharge tube of cold-cathode-ray oscillographs. It operates on a diffusion principle, and in this case permits of continuous variation of discharge-tube pressures from 0·5 mm. mercury to black vacuum at 70 kilovolts.

Other applications of a diffusion leak are indicated.

THE high-vacuum leak device described was designed for the regulation of the pressure in the discharge tube of cold-cathode high-voltage cathode-ray oscillographs. The discharge tube is continuously evacuated by a diffusion pump and air is admitted to it through some form of control valve or leak at such a rate as to maintain a pressure of about $1{\cdot}2 \times 10^{-2}$ mm. of mercury.

The leak described is based on a method used to admit small quantities of inert gas to small glass hot-cathode oscillographs. It relies for its operation on the diffusion of air through porous plugs whose porosity, whilst allowing free passage



to air, will not allow mercury to pass. A plug of this material, immersed in a cup of mercury, is connected to the apparatus under vacuum. If now a second porous plug is brought into contact with the first the mercury, owing to its high surface tension, is readily forced from between the two plugs and air is sucked through both plugs into the evacuated apparatus. The area of surface contact between the two plugs controls the rate of diffusion.

The construction finally adopted is shown in the figure. The porous plugs take the form of discs *A* and *B*, about 25 mm. in diameter. *A* is cemented into the bottom of a mercury container and is connected to the evacuated chamber. *B* is cemented flush into a rotor member *C*. A tube *D* admits air to the plug *B* and enables the rotor to be totally immersed in mercury, leaving the upper surface of *B* open to air at atmospheric pressure. It will be seen that when the rotor is in

the position shown the plug A is sealed by mercury. If the rotor is turned on its axis the two plugs will overlap; the extent by which they overlap governs the rate of diffusion, and is continuously variable. It was found advisable to make the rotor of polished rustless steel to avoid the rubbing of rust into the pores of A; and to plug the tube D with glass wool to trap grit and dust. The spring E serves to maintain the surfaces in contact. A pointer F is fitted to the rotor, and a reference scale may be attached to the container.

The magnitude of the leak can be varied by adopting different dimensions of, and materials for, the porous plugs. The plugs in use give a pressure-range from "black vacuum" to about 0·5 mm. of mercury with a discharge tube evacuated through an orifice of diameter approximately 10 mm. with a 4-stage mercury-vapour diffusion pump.

The leak can be adapted for the admission of gases to evacuated vessels by means of a double-plug rotor covering two plugs connected respectively to the gas container and to the vessel. Alternatively a flexible pipe connection may be taken to the tube D.

The thanks of the author are due to Mr W. L. Beck for suggesting the possibilities of a diffusion leak, and to the Cambridge Instrument Company Ltd. for permission to publish this note.

DISCUSSION

Prof. G. I. Finch. The leak is stated to correspond to discharge-tube pressures between black vacuum and 0·5 mm. It seems to me that this is far too wide a range, especially since it is covered by a comparatively small angle of rotation of the lever. Such a leak is definitely unsuitable for electron-diffraction and also probably for cathode-ray oscillography, but it might be possible to narrow the range sufficiently by making the upper porous plug wedge-shaped with an acute leading angle.

Author's reply. The point raised by Prof. Finch was fully appreciated in the initial stages of development. Two methods of solution presented themselves: firstly a vernier or slow-motion device for the rotor arm, and secondly a tapering plug design. Of the two, the second is admittedly the better device, but a moment's thought will show that unless the rotor is of large diameter and the plug is placed at its edge (which reduces the angle of rotation) the wedge must have curved edges, the line joining its apex to the mid-point of its base being struck about the centre of the rotor. The mechanical difficulties of this construction were considerable and were felt to be unnecessary, since leaks of the type described have now been in use, successfully, on high-speed cathode-ray oscillographs for the past few months. In this connection it should be added, however, that the oscillographs concerned employed a device, due to Prof. Finch, whereby the rate of evacuation of the discharge tube also may be continuously varied; thus the leak admitting air is not the sole means of controlling discharge-tube pressure.

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THE IMAGE-DISTORTION AND OTHER EFFECTS DUE TO THE GLASS-THICKNESSES IN LENS SYSTEMS

By H. DENNIS TAYLOR

Communicated by the Taylor-Hobson Research Department, September 23, 1933.
Read December 1, 1933.

ABSTRACT. This paper discusses the optical influence upon distortion of image, or departures from correct pictorial representation, caused by the considerable thicknesses of glass involved in the construction of high-class photographic lenses or projectors of large angular field of view. Specific instances of such influence, and the appropriate formulae for expressing the amount of such distortion, are given.

FIGURE 1*a* represents a photographic twin lens construction L_1, L_2 whose first principal point is at P_1 and second principal point at P_2. If χ is the angle which any point A_1 in the original flat picture to the left, as viewed from P_1, makes with the optic axis XX, then freedom from distortion requires that the image of A_1 shall be formed at A_2, so that $cA_2 = p_2c$ (the equivalent or conjugate focal length) multiplied by tan χ exactly.

But if the image of A_1 falls at a_2 instead of at A_2, then a_2A_2 is the linear radial displacement of the image, which in this case is towards the optic axis XX. Therefore a minus distortion is implied.

The usual method of expressing distortion is to state it in terms of R, which denotes the correct or first-order distance cA_2 from the lens axis XX, that is a_2A_2/R, or the linear displacement divided by p_2c tan χ. This is the fractional or percentage distortion, while a_2A_2 is the linear distortion. R

It is well known that proportional distortion, of the second order, increases as R^2 or as $\tan^2\chi$ and linear distortion as R^3 or as $\tan^3\chi$.

In my book *A System of Applied Optics* I advocated a method of stating and computing the various aberrations of a thick lens by the device of treating it as if it were built up of two "elements" of vanishing thickness, the first e_1 being convexo- or concavo-plane and the second e_2 being plano-convex or plano-concave, while the two elements enclose between them a parallel plate of the same glass and of a thickness t equal to the axial thickness of the solid lens. In accordance with this device the two elements e_1 and e_2 of such a thick lens may be regarded as two thin elements separated by an air space equal to t/μ, μ being the refractive index. This manner of regarding the case corresponds to the Gaussian or first-order system. t μ

Practical experience as well as further theoretical investigation has fully confirmed my confidence in the soundness of this method of treating complex lens

problems as compared with any method involving formulae which purport to apply to thick lenses as a whole. Formulae of the latter character become undesirably complex and unwieldy, owing largely to the fact that lens-thicknesses necessarily involve very serious changes in Coddington's vergency parameters α and β relating to the second surface or element.

For, if these parameters truly represent the vergencies of the rays and principal rays respectively traversing the first surface yet, if the lens be thick, they seriously fail to express in a convenient form the vergencies of the same rays traversing the second surface.

Such formulae, purporting to make use with sufficient accuracy of the vergency parameters α and β as applied to any thick lens as a whole, are undesirably complex, and this objection will be found to be still stronger when it comes to working out formulae of the third and higher orders, which are often of too much importance to be negligible, as for instance in the case of wide-angle and rapid photographic lenses used for architectural views and for aerial survey or process-copying work, in which cases it is vitally important that the photographs shall be as free as possible from any curvature or distortion affecting the images of straight lines.

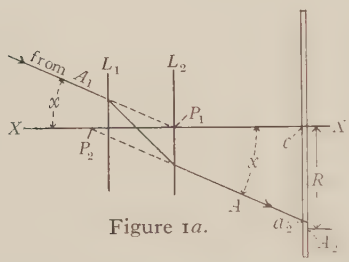


Figure 1*a*.

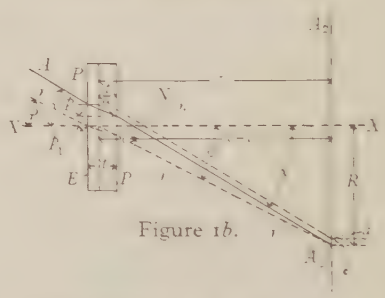


Figure 1*b*.

I will now supplement and extend the investigation of the effect of a parallel glass plate upon the distortion of images projected through it, and the formulae expressing such distortions, which are given on pages 266 to 268 of my book *A System of Applied Optics*.

Figure 1*b* represents an element E projecting an image at a distance c behind it, on to the focal plane A_2, the projection being through the parallel glass plate PP.

In this case my second formula No. 17 (page 80) gave for the linear axial aberration Nn or A_2e of the second order $\{(\mu^2-1)\ 2\mu^3\}\ a^2\ c^2$, in which μ is the refractive index, and the a^2/c^2 is obviously identical with $(pp_1\ c)^2$ of figure 1*b*, or the
χ square of the tangent of the angle of obliquity χ.

On page 267 was given the second-order formula (23) for the linear vertical or radial displacement aA_2 of figure 1*b*, which is obviously equal to the above linear-aberration formula multiplied by tan χ; therefore the formula for A_2a is $(\mu^2 - 1/2\mu^3)\ \tan^3\chi$.

c Then, if N is the point at a virtual distance c from the plate where the principal ray under first-order conditions crosses the optic axis XX, then the proportional distortion is expressed as the radial displacement A_2a in the image field A_2A_2

divided by R, the radial distance from the axis, and in this case $R = (v - c) \tan \chi$, so that the second proportional distortion is

$$t\left(\frac{\mu^2 - 1}{2\mu^3}\right) \tan^3\chi/(v - c) \tan \chi, \text{ which} = t\left(\frac{\mu^2 - 1}{2\mu^3 (v - c)}\right) \tan^2\chi \quad \ldots\ldots(1).$$

This is the formula for the second-order proportional distortion (24) given on page 268 of my book. It is obvious that if R, the radius of the field to point A_2, is to be expressed in terms of the focal length v and of the tangent ϕ which has reference to the centre of element E, then the above formula (1) becomes v, ϕ

$$t\left(\frac{\mu^2 - 1}{2\mu^3 (v)}\right) \tan^2\phi \quad \ldots\ldots(1a),$$

which gives the same result as (1). It should be noted that all such lifting or side-stepping of rays caused by their passage through the three parallel glass plates of figures 1 and 2 implies a distortion that is negative or towards the optic axis, but

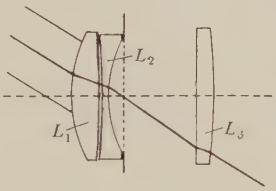


Figure 1.

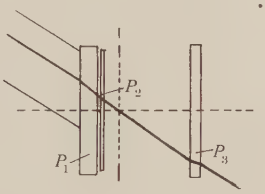


Figure 2.

that their effects upon the distortion of the final image formed by the lens system is a matter to be treated in two alternative ways, with similar results: (*a*) on the simple principle of each lens element in its turn copying through all the errors in the images presented to it*, and the other (*b*) to be explained in my next paper.

Also it is clear that any glass-thickness formulae which allow for (i) the obvious decrease of thickness, as measured by the versines of the lens curves of positive thick lenses, which takes place as the rays, when traversing the lens, recede from the lens axis, and (ii) the corresponding increase of the thickness in the case of negative lenses, will take the form of formulae of the third and higher orders only. Such formulae of the third order will be given later.

The first-order method of stating the effect of a parallel plate upon all rays traversing it obliquely is based on the assumption that

$$\tan (\text{angle of refraction}) = \tan (\text{angle of incidence})/\mu \quad \ldots\ldots(1b),$$

and therefore the effect upon any ray after emergence is simply to move it forward or side-step it parallel to itself through a distance (which is constant for all rays and is measured perpendicular to the plate) equal to $t\,(\mu - 1)/\mu$. Under such a first-order premise only, any aberration dependent upon the true relation between the angles of incidence in air and the angles of refraction within the glass is entirely ignored.

* Described in *A System of Applied Optics* (page 262 *et seq.*).

But the second- and higher-order formulae are based upon the correct law

$$\sin(\text{angle of refraction}) = \sin(\text{angle of incidence})/\mu \qquad \ldots\ldots(1c),$$

which necessarily implies aberrations from the first-order formulae dependent upon the angles of incidence of the rays upon any parallel plate of thickness t.

The formula for the linear displacement or side-stepping parallel to the optic axis Nn or A_2e in figure 1*b* of a ray (analogous to linear spherical aberration) is, when expressed by the first four orders,

$$t\left\{\frac{\mu^2-1}{2\mu^3}\tan^2\chi - \frac{3(\mu^2-1)^2}{8\mu^5}\tan^4\chi + \frac{5(\mu^2-1)^3}{16\mu^7}\tan^6\chi - \frac{35(\mu^2-1)^4}{128\mu^9}\tan^8\chi\right\} \qquad \ldots\ldots(2).$$

Then the linear side-stepping vertical to the optic axis is obviously the above formula multiplied by $\tan\chi$.

The first two terms of what is an infinite series of converging overstatements alternating with understatements of the correct aberration serve well enough for most practical purposes, while the third and fourth terms may occasionally be useful when greater exactitude is required, as in cases wherein different orders have to be separately stated in order that they may be added to the corresponding orders of the other sorts of distortion aberrations due to the same rays traversing the lens elements of the combination.

It is unfortunate however that this formula (2) involves a series of alternating + and − terms whose summed-up results diminish so gradually in accuracy that if errors of more than 5 per cent in the total are to be avoided, while the orders are kept distinct, then all the first four terms must be retained if the angle of obliquity χ exceeds 30°.

For instance, the first (or second-order term) generally exceeds the correct trigonometrical value of the linear aberration Nn or A_2e of figure 1*b* by about 44 per cent if $\chi = 45°$, by 13 per cent if $\chi = 30°$ and by 4 per cent if $\chi = 15°$, when μ is about 1·60.

It is clear that this side-stepping, as it were, of a ray after passage through a parallel glass plate, without the occurrence of any angular deviation except within the plate, distinguishes it from all other aberrations produced by the refraction of rays through lens surfaces or elements, for in the latter cases an alteration of direction or angular deviation is always imparted unless, of course, the ray traverses the centre of a thin lens or element, or passes perpendicularly through a glass surface or glass plate.

Therefore, if the effect produced in the final focal plane of an optical system has to be computed, the effects of the glass-thicknesses involved in the various lenses must be worked out separately by specially appropriate formulae, and for the purpose of working out such formulae and illustrating their application I thought I could scarcely do better than to take the case of the original (Series V) Cooke lens, whose section is given in figure 1. Figure 2 is schematic and simply represents the six element planes, each of the three pairs thereof being separated by intervals equal

to t/μ which expresses the equivalent air spaces, in accordance with the Gaussian or first-order point of view. The equivalent focal length is 10 inches.

Figure 3 represents, on a larger scale and schematically, the effects which follow in the case of the first or front lens L_1 of figure 1. Here the hypothetical parallel glass plate of the lens (whose thickness $t = 0{\cdot}299$ in.) is represented, while the thick black line E_1 on the first surface represents the first positive refracting element plane as a first-order refracting plane which has an equivalent focal length of 2·383 in. The thin black line represents the second element or refracting plane, which is also the second surface of the parallel glass plate.

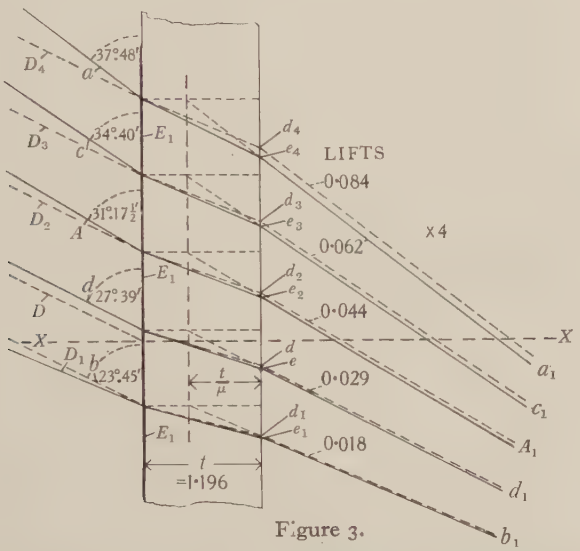


Figure 3.

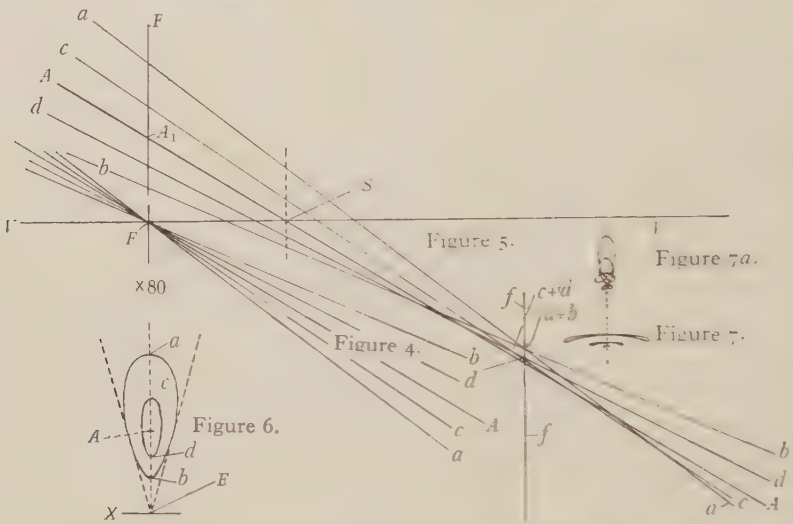
The dashed line XX is the optic axis, while the dotted entering ray D is that oblique ray from a distant object which passes through the centre of the first element, of course without being deviated, while D_1, D_2, D_3, and D_4 are also parallel to D and are the other four of the pencil of parallel rays coming from the same point in the distant object on the left. D_2 is the central ray of the oblique pencil of five rays and is regarded as the principal ray, as it passes the centre of the stop.

But after being refracted by the first element E (in accordance with first-order or Gaussian premises) these five parallel rays constituting an oblique pencil are all converged towards a point F in the principal focal plane of E_1, where the ray D continued strikes the said focal plane; and the converging lines A, d, b, c and a are continuations backwards of the rays after their refraction through the first-element plane E_1. Therefore these converging lines obviously represent the rays which impinge upon and then are all refracted through the parallel glass plate of thickness t.

The angle of obliquity with which the unrefracted ray D enters the parallel plate is here 27°, and from this the angles of obliquity of each of the other five rays after their respective refractions by E_1 are calculated. Of course these angles of obliquity refer to the perpendiculars and are restored after emergence from the glass plate.

Within the glass plate the full black lines indicate the courses of the rays according to the first-order or Gaussian assumption that tan (angle of refraction) = tan (angle of incidence)/μ, whereas the dotted lines or rays across t and their refracted continuations to the right after emergence indicate the rays as actually refracted according to the correct law that sin (angle of refraction) = sin (angle of incidence)/μ.

In this case the vertical separations or lifts shown between the same rays refracted under the first-order or Gaussian premise and under the exact law are given in inches for each ray to three decimal places, but multiplied by 4 to suit the $\times 4$ scale of the original. Also, since there is no reason why the different orders of



parallel-plate aberrations should be separately stated in this case, I thought it best to calculate the amounts of the vertical lifts l, l_1, l_2, etc. by the exact trigonometrical method, the formula being

$$l = \frac{t \tan (\text{angle of incidence})}{\mu} - t \left\{ \text{tan of angle whose sine} = \frac{\text{sine of angle of incidence}}{\mu} \right\}.$$

The vertical lift or side-stepping of these several rays, due to their passage through the first element under first-order conditions and then through the plate, brings about an effect in the focal plane of E which is illustrated in figures 4 and 5, on a 40-times' scale of magnification.

Instead of all the rays being converged by E_1 to a focus at the proper position F, figure 4, they are all lifted up parallel to themselves but by varying amounts, so that they severally cut the focal plane FF of E_1 at the points shown in figure 5 and then pass on convergently beyond the focal plane to the nearest approach to a focus, in the form of an inward pseudo-comatic flare from the axial ray AA_1 to $a - b$, figure 5.

The lift from F to A_1 of the principal ray A, figure 5, is the linear distortion of

position in the image field; the horizontal distance F to ff is the linear over-correction or throwing back of the field brought about by the glass plate. The rays refracted in the primary plane (the plane of the paper) form the inward pseudo-comatic flare at A, $c + d$ and $a + b$, figure 5, whose section in a plane perpendicular to the paper is indicated roughly in figure 7. In the secondary plane, perpendicular to the paper and the optic axis, the extreme upper and lower rays which enter the element E_1 above and below the principal ray D_2 intercross or come to focus somewhere about S, figure 5, where a pseudo-comatic flare, figure 7*a*, with its long axis lying in the plane of the diagram and pointing to the optic axis XX, is formed. Then the distance S to f between the two comatic flares is the linear astigmatism,

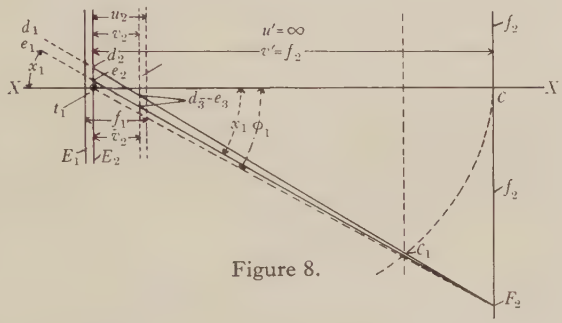


Figure 8.

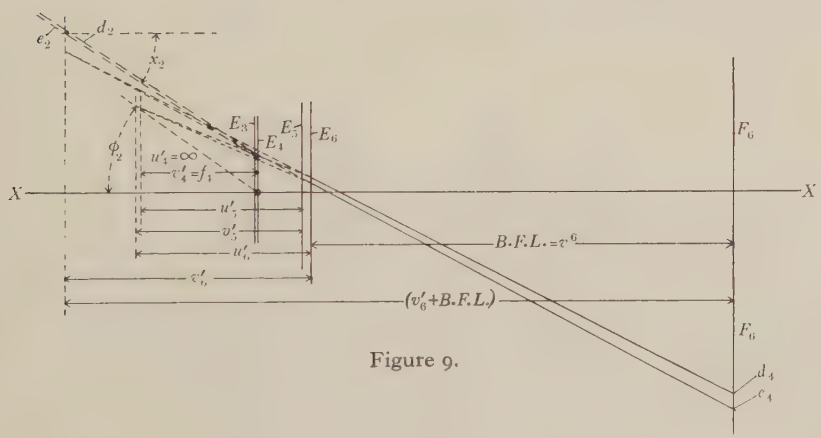


Figure 9.

which is in this case about equal to the linear over-correction of field F to $f \times 0{\cdot}625$. At smaller angles of obliquity this ratio approaches 0·66. Figure 6 represents the oval out-of-focus patch of light formed in the focal plane FF, in which A, a, b, c and d are the striking points of the principal and four other rays represented in figures 3 and 5.

It is clear that the varying parallel displacements, towards the optic axis, of the rays contained in the oblique pencil are the cause of, firstly, the displacement of the central or principal ray from its proper alignment through F, which is the linear distortion, and secondly the production of the two stigmatic and comatic formations at A and S, figure 5, separated by the horizontal stigmatic distance S to ff;

thirdly, the two horizontal distances F to f and F to S are the linear values of the two field corrections (or departures from normal field curvatures) F to ff in the primary plane and F to S in the secondary plane.

All these aberrations thus have their cause in the passage of a convergent oblique pencil through any parallel glass plate occurring in a lens system. But the aberration of the first importance, so produced, is that of the linear distortion FA_1, figure 5, while the next in importance are the accompanying comatic formations, figures 7 and 7*a*.

With regard to the distortion, I have previously given the formula (2) expressing in four orders (2nd to 5th) the linear aberration measured parallel to the optic axis and caused by the parallel plate. Also, if formula (2) is multiplied by tan χ,

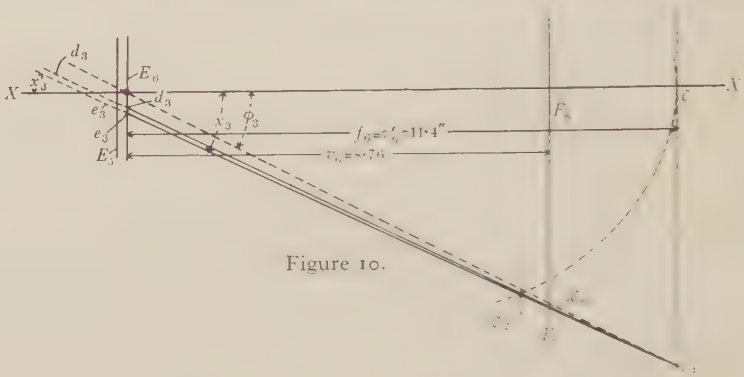


Figure 10.

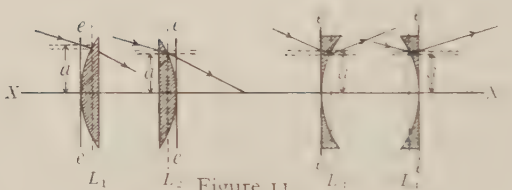


Figure 11.

we then get the radial linear distortion in the image field, and since we want the radius of the image field (or $v \tan \phi$) to be the unit, and to express the linear distortion as a percentage of that unit, then we arrive at

$$\frac{\text{Formula (2) tan } \chi}{v \tan \phi \text{ or } (v - c) \tan \chi},$$

in which $v \tan \phi$ and $(v - c) \tan \chi$ are equal alternative expressions in the fractional distortion of image. Or, more fully, we get

Fractional Distortion

$$= \frac{t}{v - c}\left\{\frac{\mu^2 - 1}{2\mu^3}\tan^2\chi - \frac{3(\mu^2-1)^2}{8\mu^5}\tan^4\chi + \frac{5(\mu^2-1)^3}{16\mu^7}\tan^6\chi - \frac{35(\mu^2-1)^4}{128\mu^9}\tan^8\chi\right\} \quad \ldots\ldots(3),$$

as in figure 1*b* in which χ is the angle of incidence on the plate of the principal ray dealt with, and not the angle of obliquity ϕ subtended by R at the centre of E, the

first of the two elements enclosing the parallel plate. It is the finely dotted ray *rr* in figure 1*b* that is inclined to the axis *XX* at the angle ϕ.

Of course it is easier to calculate the exact linear aberration of a glass plate trigonometrically by formula (1) than by the above formula (3) with its two, three or four orders, but, when one is working out the aberrations of lens systems to the degree of accuracy required for a very high standard of performance, it is of essential importance to be able to sum up all the second-order formulae which express the fractional distortions of the system and keep them separate from the third-order distortion formulae and the latter from the fourth-order, and so on.

For instance, one cannot equate to 0 or balance off second- and third-order distortions (functions of $\tan^2\chi$ and $\tan^4\chi$) for a point in the final focal plane situated, say, 33° from the axis, without necessarily implying a zone of hybrid distortion which exerts its maximum effect at an angle θ such that $\tan\theta = \tan 33^\circ \div \sqrt{2}$.

It is well known that most so-called rectilinear photographic lenses, and especially wide-angle ones, show no linear distortion at, for instance, a point in the field 27° from the centre, but show a maximum of negative distortion at an angle whose tangent is $\tan 27^\circ \div \sqrt{2}$ and a rapidly increasing positive distortion at angles greater than 27°.

So that, if no zone of hybrid distortion is to be allowed in the final image field of any particular lens combination, it is necessary to arrange that the sum of the third-order distortion formulae of the combination shall equate to 0 simultaneously with the sum of the second-order distortion formulae of the same combination also equating to 0, as nearly as possible.

DISCUSSION

Mr T. Smith. This paper should be regarded as a supplement to the author's book *A System of Applied Optics*. Perhaps the most illuminating parts of the whole discussion are figures 5, 6, 7, and 7*a*: these convey most valuable information to the designer of instruments. As the author has indicated on p. 286, the slowness with which the first few terms of the series give a sufficiently exact result is sometimes a cause of difficulty, but I think it would be helpful if the figures were supplemented by others showing the outstanding errors in terms of the axial thickness of the plate. This would give a common measure for the different cases considered.

There are often strong inducements to refrain from, or at least to delay, publication of work of this kind, but in the long run there can be little doubt that prompt publication makes for the most rapid advancement of the art of designing in which the author has achieved such notable success. We are indebted to him for the public-spirited way in which he gave us the benefit of his long and arduous investigations in his book, and for the way in which he has followed up that work in his papers. I can only hope that the gratitude we express now to Mr Taylor will lead other workers in this field to realize how very warmly contributions from them to our knowledge would be welcomed by us.

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THE MEASUREMENT OF THE INDUCTANCE OF IRON-CORED CHOKES CARRYING DIRECT CURRENT

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Communicated by Prof. T. H. Laby, July 10, 1933. *Read January* 19, 1934.

ABSTRACT. A method for measuring, by means of a reflecting dynamometer ammeter, the inductance of an iron-cored coil is described. The coil is connected in series with a capacity and the ammeter, and then the junction of the inductance and capacity and the junction of the fixed and moving coils are connected to an a.-c. supply. It is shown that for a constant applied voltage the deflection of the dynamometer ammeter is proportional to the square root of the reciprocal of the inductance. This is also true when a direct current is superimposed on the alternating current flowing through the inductance. Advantages of the method are the simplicity of the measuring instruments and of the observations taken, the small value of the alternating current used (which corresponds to the alternating current in certain applications of chokes in wireless sets), and the independence of the method within wide limits of the frequency and wave-form of the supply used.

§ 1. INTRODUCTION

The majority of existing methods of measuring the inductance of choke coils carrying direct current involve the comparison by means of a valve or thermo-couple voltmeter of the alternating potentials across the choke coil and a resistance in series with it, in a circuit carrying both direct and alternating currents. A good outline of this method is available in a paper by H. B. Dent*, and by a method of this type measurements of the inductance of iron-cored choke coils carrying direct current have been made by L. M. Turner† and by Wright and Bowditch‡.

A very ingenious method of making these measurements is given by H. M. Turner†. It uses simple laboratory apparatus, but requires a standard variable condenser and, in common with the above method, an accurate knowledge of the frequency. The present method, on the other hand, is practically independent of frequency.

* *Wireless World*, 26 August (1931).
† *Wireless Engineer*, **4**, 594 (1927).
‡ *Proc. Inst. Radio Eng.* **16**, 1559 (1928).

§ 2. PRINCIPLES OF METHOD

The actual and equivalent circuit diagrams are shown in figures 1 and 2.

For the branch of the circuit containing the choke coil let r_L ohms be the resistance, L henries the inductance and I_L amperes the alternating current through the choke; and for the branch of the circuit containing the condenser let r_C ohms be the resistance, C farads the capacity, and I_C amperes the current through it, while f cycles per second is the supply frequency, and $p = 2\pi f$. Then if r_L is negligible compared with Lp, and r_C is negligible compared with $1/Cp$, the current through the choke will lag behind the applied voltage V by 90°, and be of magnitude V/Lp, while the current through the condenser will lead the applied voltage by 90° and be of magnitude CpV.

r_L

L, I_L

r_C

C, I_C

f, p

V

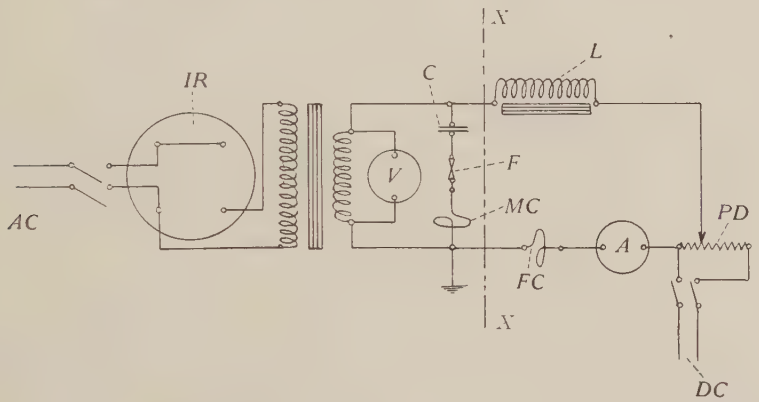

Figure 1. *AC*, a.-c. mains; *IR*, induction regulator; *V*, a.-c. voltmeter; *C*, condenser of *C* farads; *L*, choke of *L* henries; *F*, fuse; *MC*, moving coil and *FC*, fixed coil of dynamometer ammeter; *A*, d.-c. ammeter; *PD*, potential-divider; *DC*, battery.

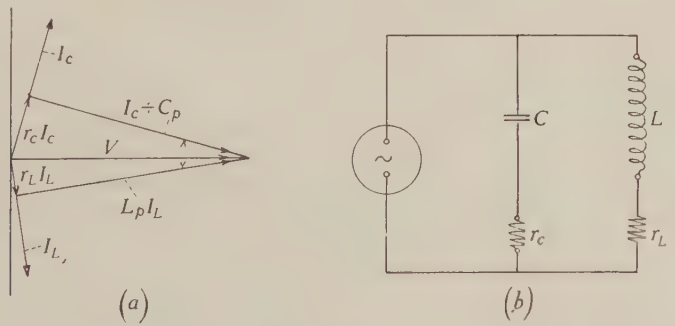

Figure 2. (*a*) Vector diagram. (*b*) Equivalent circuit.

Then the deflection d of the dynamometer ammeter, being independent of direct current flowing through the fixed coils only, will be given by

d

$$d = k\sqrt{\left(pCV \times \frac{V}{Lp}\right)} = kV\sqrt{(C/L)} \qquad \text{.....(1),}$$

where k is a calibration constant.

k

Hence $$L = k^2V^2C/d^2 \qquad \text{.....(2).}$$

This formula is independent of the frequency.

Thus for fixed voltage and capacity, the inductance is inversely proportional to the square of the dynamometer ammeter reading.

§ 3. MAGNITUDE OF ERRORS

Accuracy of formulae. Case (i) In the normal case of measurement with a suitably designed instrument $r_L\ Lp$ and Cpr_C are both small, and on consideration of the vector diagram of figure 2, the actual ammeter reading d is seen to be given by

$$d = k\sqrt{(I_C I_L \cos\phi)} = k\left[\frac{V^2}{(r_C^2 + 1/C^2p^2)(r_L^2 + L^2p^2)}(Lp/Cp + r_Cr_L)\right]^{\frac{1}{2}} \qquad \text{.....(1),}$$

therefore to the first order,

$$d^2 = (k^2V^2C/L)(1 - r_C^2p^2C^2 - r_L^2/L^2p^2 - r_Cr_LC/L).$$

Thus L is seen to be less than k^2V^2C/d^2 by the fraction

$$(r_C^2p^2C^2 + r_L^2/L^2p^2 + r_Cr_LC/L)$$

of its value.

In practice Cpr_C and $r_L\ Lp$ are both less than 0·05, so that the correction becomes less than 3 parts in 400, which is negligible.

Case (ii) If when an existing dynamometer instrument is being used as a makeshift, Cpr_C and r_L/Lp attain values so large that $(Cpr_C)^4$ or $(r_L\ Lp)^4$ are not negligible, so that the corrections discussed in case (i) are unsuitable, it is then necessary to use the equation

$$L = \tfrac{1}{2}L_1(1 + \sqrt{x}),$$

L_1 where $$L_1 = \frac{k^2V^2C}{d^2(1 + C^2p^2r_C^2)}$$

and $$x = 1 - 4[(r_L/L_1p)^2 + r_Cr_LC/L_1],$$

a formula readily derived from equation (1).

Dynamometer inductances. In the theoretical treatment given above the induct-
M ances of the moving and fixed coils and their mutual inductance M have been neglected, and normally this is quite justifiable in view of their low values as compared with the inductance to be measured; but when in the measurement of inductances with low applied a.-c. voltages I_C is increased to give a satisfactory deflection, the term MpI_C may cause an appreciable voltage-error. For example in the apparatus described in § 5 for a 50-henry choke coil $MpI_C = 10$ mV. when $V = 0{\cdot}5$ V., and under these conditions the lowest workable value of V is about 1 V. For low a.-c. voltages it was found convenient to place between the portions of the circuit to the left and right of XX, figure 1, a 100:1 step-down transformer which enabled 200 volts to be applied to the condenser with 2 volts across the inductance. This permits a smaller condenser to be used and the formula $L = k^2V^2C/100d^2$ holds, the elaborate corrections of case (ii) being now un-

necessary; and further the extra transformer is used under light-load conditions, so that corrections for its voltage-ratio and phase-angle are negligible.

The d.-c. potentiometer. Normally the potentiometer-resistance is variable and is included in r_L, but in case (i) the corrections containing r_L as a factor are negligible or only first-order terms, and so the changes in r_L have no effect. In case (ii) it is necessary to keep constant the potentiometer resistance included in r_L, and this is achieved within close limits by keeping constant the potentiometer resistance in the inductance circuit, keeping it low in value, and varying the d.-c. potential-drop across it by means of battery taps and resistance in the battery circuit. When low-impedance chokes are being measured the potentiometer resistance may be made very small by using an accumulator battery with terminal taps and a low resistance potentiometer across the end cell, all in series with the inductance to be measured.

It will now be shown that if a dynamometer voltmeter is used, any harmonics present in the secondary voltage will produce substantially no effect on the voltmeter and ammeter readings.

Voltmeter reading. If the instantaneous voltage across the secondary of the transformer is $\bar{V}$, where $\bar{V}$

$$\bar{V} = V_0 + \sqrt{2}\,[V_1 \sin pt + V_2 \sin (2pt + \phi_2) + \ldots + V_r \sin (rpt + \phi_r)]$$

and V_0 is the d.-c. voltage-drop in the transformer winding;

V_r the root-mean-square value of rth voltage harmonic; and V_r, r

ϕ_r the phase difference between $\bar{V}_r$ and $\bar{V}_1$, ϕ_r

then the reading V of the dynamometer voltmeter will be given by V

$$V = (V_0^2 + V_1^2 + \ldots + V_r^2 + \ldots)^{\frac{1}{2}}$$

and the effect of V_0 may generally be neglected; for if V_m is the maximum d.-c. voltage applied to the circuit by the potentiometer and r, R the resistances of the transformer secondary and the circuit connected across the potentiometer, then $V_0 = (r/R)\,V_m$, which in practice can readily be made less than 0·05 V., by the use of a transformer with a low-resistance secondary. Since this value of V_0 affects the value of V only to the extent of $\frac{1}{2}(0{\cdot}05)^2 V$, or 1/800 V_1, it is negligible. V_m r, R

Ammeter reading. With the applied voltage $\bar{V}$ the current through the inductance will, to the first order, be

$$\frac{V_m}{R} + \frac{V_1}{Lp}\sin\left(pt - \frac{\pi}{2}\right) + \frac{V_2}{2Lp}\sin\left(2pt + \phi_2 - \frac{\pi}{2}\right) + \ldots + \frac{V_r}{rLp}\sin\left(rpt + \phi_r - \frac{\pi}{2}\right)$$

while the current through the condenser will be

$$CpV_1 \sin\left(pt + \frac{\pi}{2}\right) + 2CpV_2 \sin\left(2pt + \phi_2 + \frac{\pi}{2}\right) + \ldots + rCpV_r \sin\left(rpt + \phi_r + \frac{\pi}{2}\right)$$

and these currents will produce a reading of the dynamometer ammeter equal to

$$k\left\{\frac{C}{L}(V_1^2 + V_2^2 + \ldots + V_r^2 + \ldots)\right\}^{\frac{1}{2}}.$$

Thus if a dynamometer voltmeter is used and the secondary voltage V is kept constant, the effect of voltage harmonics will cancel out.

§ 4. EXPERIMENTAL DETAILS

Apparatus. The following instruments are required: (i) A permanent-magnet moving-coil ammeter, to measure the direct current. (ii) A dynamometer ammeter whose fixed coil is capable of carrying the maximum direct current through the choke, and whose moving coil should preferably have a current capacity greater than that of full-scale reading. (iii) A transformer to give the required voltage V, with an isolated secondary to prevent the d.-c. battery from being at a high potential above earth. (iv) An a.-c. voltmeter, preferably of the dynamometer type, with a sensitive movement. (v) An induction regulator in the transformer primary to keep the secondary voltage V constant. Although this is not absolutely necessary it greatly simplifies the method, as regards both the taking of readings and the working out of the results. An induction regulator gives smoother adjustment than primary resistances. (vi) A potential-divider to introduce the direct current. (vii) A condenser of good insulation resistance and known capacity C farads, to be selected to suit the measurement required. This apparatus is connected as shown in figure 1.

Design of dynamometer instrument. In all cases the fixed coils must have a current capacity at least equal to D, the largest direct current to pass through the choke coils to be tested. Then for maximum range of inductance-measurement it is advisable to have the current capacity of the moving system at least $0{\cdot}1\,D$ and to select the suspension so that the full-scale deflection is approximately $0{\cdot}01\,D$. In conjunction with this, it is advisable to keep the resistances of the fixed and moving coil systems as low as possible.

For wireless applications it was found convenient to have D equal to 500 mA., with a phosphor-bronze suspension to carry the moving-coil current.

Setting up the apparatus. To do this to best advantage, it is necessary to know the range of use of the dynamometer ammeter, which is decided as follows. If the satisfactory range of the ammeter scale is between i_0 and i_1, then I_L and I_C are limited by the relation $i_1^2 > I_C I_L > i_0^2$ and the current-carrying capacity of the fixed and moving coils. Thus if V volts a.-c. be the voltage at which the inductance is required, and l a rough estimate (within several hundred per cent) of the inductance, a guide to the selection of a suitable value of C is given by

$$C = \frac{l}{V^2}\left(\frac{i_0 + i_1}{2}\right)^2.$$

Procedure. A value of C having been selected and the resistance of the choke measured, the apparatus is connected as shown in figure 1. The direct current is set to the required value as indicated by the d.-c. ammeter, an a.-c. voltage is switched on and adjusted to the value V by means of the induction regulator, and the reading d of the dynamometer ammeter is taken. From these readings a value of L is calculated according to the formula $L = k^2 V^2 C\, d^2$, and the values of $r_L Lp$ and Cpr_C are determined, enabling us to select the method of working out the results.

§ 5. EXPERIMENTAL RESULTS

A number of graphs of inductance were plotted against direct-current-strengths for different applied a.-c. voltages, and were found to agree with those of L. M. Turner. The results were all taken with a direct-reading dynamometer milliammeter giving a full-scale deflection with 6 mA. It had fixed and moving coils of 89 and 85 ohms respectively, the current capacity of the fixed coils being approximately 500 mA., while that of the moving system was limited by the suspension to about 50 mA. The resistance of the instrument is rather inconveniently high for the measurement of low-impedance choke coils; but this defect makes it suitable for illustrating the procedure when large corrections occur, an event which would but rarely arise with a specially designed instrument.

For the following observations a.-c. mains with p equal to 314·2 were used as the source of power, and r_L was equal to 85 ohms throughout.

Table 1. These results were taken on a commercial choke coil rated at 230 henries at 5 mA. d.c. Its resistance was 3800 Ω.,

$$V \text{ was } 85 \text{ V.}, \quad C \text{ was } 0{\cdot}292\,\mu\text{F.}, \quad \text{and} \quad r_L \text{ was } 4000\,\Omega.$$

Hence $Cpr_C = 0{\cdot}0078$ and $r_L/Lp = 0{\cdot}056$, so that both are negligible.

$$V^2C = 85^2 \times 0{\cdot}292 \times 10^{-6} = 2110 \times 10^{-6}.$$

Table 1

Direct current (mA.)	Dynamometer reading I (mA.)	L, or $2110/I^2$
0	2·215	431
4	2·665	296
8	2·970	240

Table 2 relates to a stalloy choke coil with no air gap, rated at 20 H. with no direct current. Its resistance was 84 Ω., r_L was 180 Ω., V was 2·01 V., and C was 19·4μF., so that $V^2C = 78{\cdot}45 \times 10^{-6}$, and $Cpr_C = 0{\cdot}518$.

Thus $(Cpr_C)^4 = 0{\cdot}072$, and the results must therefore be calculated according to case (ii) of § 3.

$$V^2/\{1 + (Cpr_C)^2\} = 61{\cdot}8 \times 10^{-6}; \quad 4r_C r_L C = 1{\cdot}19; \quad \text{and} \quad 4\,(r_L/p)^2 = 1{\cdot}31.$$

Table 2

Direct current (mA.)	Dynamometer reading (mA.)	L_1, or $61{\cdot}8/I^2$	$\frac{1{\cdot}19}{L_1}$	$\frac{1{\cdot}31}{L_1^2}$	$\frac{1{\cdot}19}{L_1} + \frac{1{\cdot}31}{L_1^2}$	x	$1 + \sqrt{x}$	L, or $\frac{1}{2}L_1(1 + \sqrt{x})$
0	1·575	24·9	0·048	0·002	0·050	0·950	1·975	24·6
10	2·230	12·4	0·096	0·008	0·104	0·896	1·945	12·1
50	3·600	4·76	0·250	0·058	0·308	0·692	1·832	4·36
100	4·615	2·90	0·411	0·156	0·567	0·433	1·657	2·40

§ 6. ADVANTAGES OF THE METHOD

The dynamometer ammeter is extremely simple and is readily made in any laboratory, and the calibration can be simply carried out with reversed direct current; the method is independent, within wide limits, of wave-form and frequency; the range of measurements covered by any one instrument is large; measurements may be made with very small a.-c. currents passing through the choke coil—down to 0·04 mA. for the dynamometer used in the experimental work; and only one reading has to be recorded for each observation, the dynamometer reading being taken when the voltage is brought to the correct value, while small supply-frequency variations are neglected.

§ 7. ACKNOWLEDGMENTS

This work was carried out in the Electrical Engineering Laboratory of the University of Melbourne under the terms of the Robert Bage Memorial Scholarship Award for 1933.

I wish to thank Prof. T. H. Laby, M.A., D.Sc., F.R.S., for his advice and assistance in the arrangement of the paper.

DISCUSSION

Dr D. Owen. The author has applied the electrodynamometer to the measurement of the inductance of iron-cored chokes carrying direct current, a condition which is usual in many valve circuits. The use of the condenser in series with one of the coils of the instrument surmounts the difficulty arising from the presence of the direct current.

It is, however, misleading to term the instrument employed a dynamometer ammeter, since the two coils are not in series. In the section on the design of the dynamometer the question of the most suitable windings for the two coils might with advantage have been treated. It is fairly clear that for the present application the corrections required, when the formula is used in its simplest form, will be diminished by making the resistance of the coil in series with the condenser as low as possible, while the second (fixed) coil may have many more turns and a much larger resistance, in order to retain the desired sensibility.

In regard to the experimental data quoted it should be made clear that it is the a.-c. resistance that is required in applying the appropriate corrections to the simple formula (1). The importance of these corrections is consequently underestimated in the examples used in the paper.

It may be noted that the error which in certain cases may arise from the induced voltage injected by one coil into the other can be eliminated by taking a second reading with the connections to one of the coils reversed. But it is possible to avoid this source of error altogether by modifying the method so that it becomes a null one. It is hoped to investigate this possibility by experiment.

Author's reply. Dr Owen's criticism of the use of the term "dynamometer ammeter" to specify the instrument is quite justified and the word could be replaced

by "electrodynamometer", for in some applications of the method a dynamometer wattmeter was used and a ratio of fixed to moving coil currents of 10,000 to 1 arose. The design of the instrument is in no way critical, and any dynamometer with a sensitive movement, a low-resistance moving-coil system and large number of turns on the fixed-coil system will cover a large range of measurements. The size of winding-wire is suitably determined by the current capacities suggested, and the diameter of the coils is selected according to whether spring or single-wire suspension is adopted. The moving-coil system is made up of as many turns and as low a resistance as possible, 50 ohms being a suitable value; then with I_0 maximum flowing through the moving coils the a.-c. ampere-turns of the fixed coils must be sufficient to give a suitable deflection with a current equal to V/Lp, minimum, flowing through them.

These ampere-turns are most simply found by mounting the moving-coil system and finding the current required through a few heavy turns mounted on the fixed-coil formers, to give the desired deflection. It can then be considered whether it would be advisable to use a more sensitive suspension with fewer fixed-coil turns, and care should be taken to keep the mutual inductance between the moving and fixed coils as low as is compatible with a reasonably robust suspension.

The error in taking r_L as the d.-c. instead of the a.-c. resistance of the choke coil will generally be of no consequence, for on low frequencies the approximation is satisfactory and r_L/Lp decreases in importance with frequency; moreover r_L/Lp is in general quite a small correction term. However, when a high a.-c. voltage is applied across a low-impedance choke coil the possibility of iron losses increasing the equivalent r_L should be considered.

The suggested method of eliminating the effect of voltage induced in the moving coil by the fixed, in common with a number of compensation methods tried out, suffers from the disadvantage that on reversal of the connection to one of the instrument coils, the deflection is reversed. This necessitates a centre zero, which in reflecting instruments gives trouble owing to zero-creep.

I have given considerable thought to Dr Owen's suggestion that the method could be made a null one, and only two ways occur to me in which this could be done. (1) Place across the inductance a variable condenser and a variable resistance in series, and tune so that the parallel impedance is resistive. This method would be far too insensitive, for even though the reactive component of the impedance increases rapidly on either side of resonance, the impedance remains very large owing to the high L/C ratio usually met with in its range of use. (2) Use double-wound fixed coils, one half of the winding being in series with inductance and d.-c. potentiometer as in the circuit diagram, the other half in series with a variable condenser and resistance, and the two circuits being in parallel. This arrangement can be tuned to zero deflection, but the method will be no longer independent of wave-form and frequency, it will not be so convenient to use, and the apparatus required will be more elaborate. This plan should, however, be valuable for tuning low-frequency circuits under working conditions, for the iron in the choke coil can be brought to the same saturation as that in the plate circuit of a valve.

REVIEWS OF BOOKS

Theory of Thermionic Vacuum Tubes, by E. L. CHAFFEE, Professor of Physics, Harvard University. Pp. 652 + xxiii, with 347 diagrams and 6 plates. (McGraw-Hill, 1933.) Price $6.00.

Great progress in the design and uses of the many forms of the thermionic tube is due to be recorded since the excellent treatise on this subject by Van der Bijl, which appeared in 1920 under the auspices of the same publishers. The author announces in his preface that the present volume is written primarily as a text-book, though he hopes it may serve also as a reference book. In view of its formidable size this must seem somewhat disconcerting to the student of physics or of electrical engineering, to whom the subject can rank only as one of many to be mastered, as well as may be, in the course of a normal university training. This feeling, moreover, will be accentuated when he learns that owing to the mass of material involved only the theory of vacuum tubes at low power is here included, and that the remainder, including the theory of power amplifiers, oscillators, gas-content tubes, rectifiers, etc., is held over for treatment in a subsequent volume.

But whether as text-book or for reference, this book will be found to contain an eminently sound treatment of the characteristic properties and multiform applications of hard thermionic tubes, and will be appreciated by all interested in the design and precise functioning of these devices. The first five chapters give an introductory account of the physics of the valve, of the various types of electronic emission, and of emitting materials. After a brief chapter allotted to nomenclature follow three chapters containing a treatment, partly analytical and partly graphical, of the actions within the triode valve, with its equivalent circuits and its six coefficients and the modes, static and dynamic, of their measurement. A chapter is given to the effect of the presence of slight traces of gas on the characteristics.

In all fifteen chapters are devoted to the triode, first as detector and then as amplifier both at audio- and radio-frequencies. The various types of multi-stage amplification, with and without regenerative action, at small amplitudes and at large amplitudes, are in turn dealt with. The modes of energy-interchange between grid and plate circuits of a triode are analyzed, as well as the means of nullifying or "neutralizing" this effect. The same combination of analytic and graphic treatment is applied in a final chapter on tetrodes and pentodes.

The book is well written and produced, the illustrations are abundant and excellent, and copious references are given to the literature of the subject. Space models as well as the usual two-dimensional characteristics are discussed and represented in clear photographs. A few points of criticism may be mentioned. First, the absence of reference to the magnetron. Next the brevity of treatment of the photoelectric effect, the normal effect only being described in the text, whilst the illustrative curves are confined to the selective effect. Again, the classification of amplifiers into classes A, B and C seems likely to cause confusion, in view of the different terminology largely in use now. It must, in conclusion, be again pointed out that the ground covered in this volume is only part, probably only half, of that indicated in the title. It would surely have been better to have made this fact quite explicit, by publishing the present book as volume 1 of the complete work.

D. O.

Physics for Medical Students. A supplementary text-book, by J. S. ROGERS, B.A., M.Sc., F.Inst.P. Edited by T. H. LABY, F.R.S. Pp. 197 (including two appendices) and index. 71 figures (including 6 half-plates). (Melbourne University Press in association with Oxford University Press.) 11*s.* 6*d.*

The title of this book rightly indicates that it has not been written to meet the needs of medical students who are reading for the pre-registration examination in physics. We judge that the writer had in mind medical students with an aptitude for physics who might wish for guidance in certain physical subjects that bear on general physiology and medical practice. Such students will find in the book under review brief but lucidly written chapters on certain aspects of so-called bio-physics (e.g. those on hydrogen ion concentration, osmosis, and colloids, though, strange to say, there is no chapter on surface phenomena) and of human physiology (e.g. those on blood pressure, the energy relations of the human body, the resonance theory of hearing, and the human eye) and on other physical topics (e.g. ultra-violet light, high frequency currents, X-rays, radio-activity, and the theory of the microscope) that concern the medical man. The book begins with a sketch of the history of physics. M. T.

The Calculus of Finite Differences, by L. M. MILNE-THOMSON. Pp. xxiii + 558. (London: Macmillan & Co.) 30*s.* net.

To those who have not kept in touch with recent work in the theory of finite differences this book will come as a revelation. It falls roughly into two parts, seven chapters on interpolation and numerical work, with an account of the Bernoulli polynomials and certain generalizations of them, and ten chapters on the mathematics of the finite-difference equation.

The first section is well done, as must be expected from the author of *Standard Tables of Square Roots* and *Jacobi's Elliptic Functions*, and part-author of *Standard Four-Figure Tables*. It is, of course, only one of at least three good presentations of the subject in English, but it differs from the usual treatise in giving much more attention, with every form of interpolation formula, to tables which proceed by unequal increments of the argument.

The second part opens with a section on the summation problem for the equation of first order and degree, in which Nörlund's method is adopted. This is followed by a fascinating chapter on the difference equation of the first order, which Boole would have revelled in could he have lived to read it. It contains among other results the general solution to the complete linear equation of the first order, and a discussion of the theory of the exact equation of any degree. This chapter and the next two, on equations of the first degree but any order, cover practically all the ground that was known to Boole, though it is now presented in a rigorous form, in striking contrast to the vague arguments with which he had to be satisfied.

Chapter 14 deals with Milne-Thomson's extension of the operational method, first published in 1932. From this point to the end the book is not easy reading, largely owing to the strain on the memory due to the continual symbolism. With such a conspectus before one, it is possible to realize the present position of the theory of the difference equation clearly. It is surprising how much of the material is met with in treatises on analysis or on infinite series, and interesting to notice that practically all the processes familiar in the case of ordinary differential equations, except the application of group theory, now have their counterparts in the calculus of differences. Partial difference equations receive little consideration and apparently our knowledge of them has hardly increased in the last forty years.

The book under review is certainly not an elementary introduction to its subject, and should not be read by those whose knowledge of differential equations is less than can be obtained from the study of such a book as that by Ince. For those who are seriously hoping to learn something of the subject, there are examples in copious supply at the ends of almost all the chapters. Since examples are an essential aid to the understanding, as well as to the mental retention, of any book on mathematics, it is difficult to see why they are not invariably provided; and since they are so often absent it is only right to signalize the fact when they are present. J. H. A.

Physical Constants, by W. H. J. Childs, B.Sc., Ph.D. Pp. viii + 77 with 7 nomograms. (London: Methuen & Co., Ltd.) 2*s*. 6*d*. net.

This book of tables of physical constants is an admirable example of bovrilization; it gives more useful information to the square centimetre than any other book of its kind with which the reviewer is acquainted. The contents are well arranged and selected, and, as far as the reviewer has tested them, accurate and up to date. An interesting feature is the use made of nomograms. The book should have a wide circulation; it may be strongly recommended. A. F.